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Studies on five-dimensional relativity theory

By C. V. Jonsson

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Summary

In the generalized five-dimensional theory, developed by Klein, the coordinate system can be chosen so that the five components γ_{00}, γ_{m0} of the fundamental tensor $\gamma_{\mu\nu}$ are independent of the fifth coordinate x^0. In the first part of this paper the rigorous field equations in such a coordinate system are derived, and the five-dimensional energy-momentum tensor is calculated. In the second part the field equations obtained by linearization are examined, and it is shown that the equations then describe fields of spin $2\hbar$. The quantization of these fields is performed, and the interaction with spinor particles is taken into consideration.

Introduction

In the generalized five-dimensional relativity theory [1, 2], whose foundations have recently been the object of a revision by O. Klein [3], there is introduced, together with four coordinates x^1, x^2, x^3, x^4, describing the space-time continuum, a fifth coordinate x^0, conjugate to the electric charge. It is a characteristic feature of this generalized theory, that all field quantities have to be periodic functions of the fifth coordinate, which in this respect is distinguished from the other coordinates.

The five-dimensional world is described by the line element[1]

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu\, dx^\nu,$$

and it is the ultimate purpose of the theory to solve the field equations determining the behaviour of the fundamental tensor $\gamma_{\mu\nu}$, which, due to the nonlinearity of the equations, is, of course, an extremely complicated problem. The nearest task must be an attempt to push forward as long as possible against the solution, by methods, analogous to those used in the corresponding four-dimensional problems.

The field equations for the generalized field are obtained from a variation principle in close analogy to the derivation of the equations describing the gravitational field in current four-dimensional theory. These five-dimensional field equations may be given in a form which facilitates their interpretation and their comparison with ordinary formulas, if the coordinate systems are specialized in such a manner that five of the fifteen components of the symmetric fundamental tensor are independent of the fifth coordinate. The field equations thus obtained are derived explicitly in this paper through direct calculation of the components of the five-dimensional curvature tensor.[2]

As an attempt to obtain further knowledge about the consequences of the five-dimensional field theory, Klein [2] examined the approximate equations obtained by linearization, in close correspondance to the treatment of gravitational waves performed by Einstein [4] in the early days of general four-dimensional relativity theory. It was pointed out by Klein that the linearization leads to wave equations of the meson type, the generalized five-dimensional theory thus including charged fields, corresponding to particles of non-vanishing rest-mass, together with the gravitational and electro-magnetic fields.

This linear approximation is more closely investigated in this paper, and it is shown that these wave equations describe particles of spin $2\hbar$, according to the theory of Fierz [5], the corresponding particles being therefore a sort of heavy charged gravitational quanta. If we go to the next approximation it is found, moreover, that the equations obtained describe particles of spin $2\hbar$ in interaction with photons.

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[1] See Notations, p. 89.

[2] The results of these calculations are found in §§ 5–8.

Matter is introduced into the theory as spinor particles of half integer spin[1], described by means of a generalized DIRAC wave equation analogous to that given for the behaviour of DIRAC particles in gravitational fields [6–10]. It is shown that the interaction terms obtained in the linear approximation are essentially of the form known from the meson pair theories.

In connection with this investigation the quantization of the free field of spin $2\hbar$ is carried out explicitly by a direct calculation, which is shown to be equivalent with the general method indicated by FIERZ.

In publishing this thesis I wish to express my sincere gratitude to my teacher, Prof. O. KLEIN, for the suggestion of the study of these problems and for his inspiring guidance throughout the course of the work. My thanks are also due to Dr. B. BRUNO for much stimulating criticism and many valuable advices.

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Notations

Greek indices $\mu, \nu, \varrho, \sigma, \ldots$ are always[2] going from 0 to 4, Italics $m, n, r, s, \ldots$ from 1 to 4, and Latin indices $\mathrm{m}, \mathrm{n}, \mathrm{r}, \mathrm{s}, \ldots$ from 1 to 3. Following the EINSTEIN summation convention, we are always summing over indices appearing twice in the same term. The tensor corresponding to the four-dimensional fundamental tensor g_{mn} is denoted by $\gamma_{\mu\nu}$. Greek capitals $\Lambda, M, P, \Theta, \ldots$ are used to indicate quantities in five-dimensional theory, which are built up in the same way as the corresponding four-dimensional quantities $L, M, R, T, \ldots$, thus, e. g. $P_{\mu\nu}$ is the tensor obtained, if all quantities in the expression for the four-dimensional curvature tensor R_{mn} are replaced by their five-dimensional analogues.

The absolute values of the determinants $|\gamma_{\mu\nu}|$ and $|g_{mn}|$ are denoted by γ and g respectively. The sign $\breve{}$ indicates tensor densities, thus, e. g. $\breve{\Lambda}$ and $\breve{L}$ are the LAGRANGIAN densities $\Lambda\sqrt{\gamma}$ and $L\sqrt{g}$. For the volume element we introduce the abbreviations

$$d\mathrm{v} = dx^1\,dx^2\,dx^3, \quad dv = dx^1\,dx^2\,dx^3\,dx^4, \quad d\vartheta = dx^0\,dx^1\,dx^2\,dx^3\,dx^4.$$

The constant β is defined by the relation

$$\frac{\beta^2}{2} = \frac{8\pi K}{c^4}, \tag{1 a}$$

where K is NEWTON's gravitational constant,

$$K = 6.7 \cdot 10^{-8}\ \mathrm{g}^{-1}\,\mathrm{cm}^3\,\mathrm{sec}^{-2},$$

---

[1] In this paper, the word "matter" is solely used in this sense.

[2] In the study of the linear approximation, the coordinate x^0 is replaced by another coordinate x^5 defined by $x^5 = x^0\sqrt{\alpha_0}$, α_0 being a large constant. Greek indices are then going from 1 to 5.

which gives

$$\beta = 2.0 \cdot 10^{-24}\ \mathrm{cm}^{-\frac{1}{2}}\,\mathrm{g}^{-\frac{1}{2}}\,\mathrm{sec}\,(\mathrm{dyn}^{-\frac{1}{2}}). \tag{1 b}$$

The five-dimensional three-index symbols are denoted by $\Gamma^{\varrho}_{\mu\nu}$ and $\Gamma_{\varrho,\,\mu\nu}$,

$$\Gamma^{\varrho}_{\mu\nu} = \gamma^{\varrho\sigma}\,\Gamma_{\sigma,\,\mu\nu} = \tfrac{1}{2}\,\gamma^{\varrho\sigma}\left(\frac{\partial\gamma_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial\gamma_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial\gamma_{\mu\nu}}{\partial x^{\sigma}}\right), \tag{2 a}$$

and the corresponding four-dimensional CHRISTOFFEL symbols by G^{r}_{mn} and $G_{r,\,mn}$[1],

$$G^{r}_{mn} = g^{rs}\,G_{s,\,mn} = \tfrac{1}{2}\,g^{rs}\left(\frac{\partial g_{ms}}{\partial x^{n}} + \frac{\partial g_{ns}}{\partial x^{m}} - \frac{\partial g_{mn}}{\partial x^{s}}\right). \tag{2 b}$$

The sign $\sim$ imports that, in a four-dimensional expression, all derivatives $\dfrac{\partial}{\partial x^{m}}$ have to be replaced by the operators

$$\partial_m = \frac{\partial}{\partial x^{m}} - \beta A_m \frac{\partial}{\partial x^{0}},$$

where A_m is the electro-magnetic potential vector. Thus,

$$\tilde{G}^{r}_{mn} = g^{rs}\,\tilde{G}_{s,\,mn} = \tfrac{1}{2}\,g^{rs}\,(\partial_n g_{ms} + \partial_m g_{ns} - \partial_s g_{mn}). \tag{2 c}$$

As to the definition of the symbols $\tilde{\Gamma}_{\mu,\,\varrho\sigma}$, see § 4.

PART I

The rigorous field equations

§ 1. The variation principle

The metric is described by the line element

$$d\sigma^2 = \gamma_{\mu\nu}\,dx^{\mu}\,dx^{\nu}, \tag{3}$$

and the field equations are obtained by the variation

$$\delta \int \left(\breve{\Lambda} + \frac{\beta^2}{2}\breve{M}\right) d\vartheta = 0, \tag{4}$$

the $\gamma_{\mu\nu}$ being kept constant at the limits. M is the LAGRANGIAN belonging to matter, and the LAGRANGIAN of the field is given by the expression

$$\Lambda = \gamma^{\mu\nu}\,(\Gamma^{\varrho}_{\mu\sigma}\,\Gamma^{\sigma}_{\nu\varrho} - \Gamma^{\varrho}_{\mu\nu}\,\Gamma^{\sigma}_{\sigma\varrho}), \tag{5}$$

---

[1] We deviate from the notations $\{mn,\,r\}$ and $[mn,\,r]$ generally used in the literature, in order to bear in mind the analogy between $\Gamma^{\varrho}_{\mu\nu}$ and G^{r}_{mn}.

which is the exact analogue to the LAGRANGIAN of the gravitational field in ordinary four-dimensional theory.

The variation principle leads to the fifteen field equations

$$\Gamma_{\mu\nu} = -\tfrac{1}{2}\beta^2\,\Theta_{\mu\nu}, \tag{6}$$

where $\Theta_{\mu\nu}$ is a five-dimensional generalization of the energy-momentum tensor T_{mn}, and where

$$\Gamma_{\mu\nu} = P_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}P \tag{7}$$

with

$$P = \gamma^{\varrho\sigma}P_{\varrho\sigma} \tag{8}$$

is the tensor, corresponding to the four-tensor

$$G_{mn} = R_{mn} - \tfrac{1}{2}g_{mn}R,$$

R_{mn} denoting the curvature tensor

$$R_{mn} = G^{r}_{ms}G^{s}_{nr} + \frac{\partial^2}{\partial x^m\,\partial x^n}\log\sqrt{g} - \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^r}\left(\sqrt{g}\,G^{r}_{mn}\right). \tag{9}$$

It follows from (9) that the tensor components $P_{\mu\nu}$ are given by the expressions

$$P_{\mu\nu} = \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\varrho} + \frac{\partial^2}{\partial x^\mu\,\partial x^\nu}\log\sqrt{\gamma} - \frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^\varrho}\left(\sqrt{\gamma}\,\Gamma^{\varrho}_{\mu\nu}\right). \tag{10}$$

§ 2. Specialization of the coordinate system

As has been pointed out by KLEIN, we get the connection with classical four-dimensional field theory in the most obvious way, if the coordinate systems are so chosen that the five tensor components $\gamma_{\mu 0}$ are independent of x^0,

$$\frac{\partial\gamma_{00}}{\partial x^0} = 0, \quad \frac{\partial\gamma_{m0}}{\partial x^0} = 0, \tag{11 a}$$

throughout all space. This is always possible, as may be seen from the following considerations.

Presume a coordinate system x^0, x^1, x^2, x^3, x^4, where (11) are not satisfied, and perform a coordinate transformation

$$x^\varrho = \xi^\varrho(x'^\nu) \qquad [\xi^\varrho(x'^\nu) = \xi^\varrho(x'^0, x'^1, x'^2, x'^3, x'^4)], \tag{12}$$

thereby introducing a new coordinate system $x'^0, x'^1, x'^2, x'^3, x'^4$. The components of the fundamental tensor go over into

$$\gamma'_{\mu\nu}(x'^\tau) = \frac{\partial\xi^\varrho}{\partial x'^\mu}\,\frac{\partial\xi^\sigma}{\partial x'^\nu}\,\gamma_{\varrho\sigma}(x^\tau),$$

and thus we have

$$\frac{\partial \gamma'_{\mu 0}}{\partial x'^0} = \frac{\partial \xi^\varrho}{\partial x'^\mu} \frac{\partial \xi^\sigma}{\partial x'^0} \frac{\partial \xi^\lambda}{\partial x'^0} \frac{\partial \gamma_{\varrho\sigma}}{\partial \xi^\lambda} + \frac{\partial^2 \xi^\varrho}{\partial x'^\mu \partial x'^0} \frac{\partial \xi^\sigma}{\partial x'^0} \gamma_{\varrho\sigma} + \frac{\partial \xi^\varrho}{\partial x'^\mu} \frac{\partial^2 \xi^\sigma}{\partial x'^{02}} \gamma_{\varrho\sigma}.$$

Separating $\mu = 0$ and $\mu = m$, we find that the equations (11),

$$\frac{\partial \gamma'_{00}}{\partial x'^0} = 0, \quad \frac{\partial \gamma'_{m 0}}{\partial x'^0} = 0, \tag{11b}$$

may be satisfied, if it is possible to solve the system of differential equations

$$\begin{cases} \dfrac{\partial^2 \xi^\sigma}{\partial x'^{02}} \gamma_{\varrho\sigma} \dfrac{\partial \xi^\varrho}{\partial x'^0} = -\dfrac{1}{2} \dfrac{\partial \xi^\sigma}{\partial x'^0} \dfrac{\partial \xi^\varrho}{\partial x'^0} \dfrac{\partial \xi^\lambda}{\partial x'^0} \dfrac{\partial \gamma_{\varrho\sigma}}{\partial \xi^\lambda} \\[2ex] \dfrac{\partial^2 \xi^\sigma}{\partial x'^{02}} \gamma_{\varrho\sigma} \dfrac{\partial \xi^\varrho}{\partial x'^m} = -\dfrac{\partial \xi^\varrho}{\partial x'^m} \dfrac{\partial \xi^\sigma}{\partial x'^0} \dfrac{\partial \xi^\lambda}{\partial x'^0} \dfrac{\partial \gamma_{\varrho\sigma}}{\partial \xi^\lambda} - \dfrac{\partial^2 \xi^\varrho}{\partial x'^0 \partial x'^m} \gamma_{\varrho\sigma} \dfrac{\partial \xi^\sigma}{\partial x'^0} \end{cases} \tag{13}$$

We choose a certain fixed value of x'^0 and introduce the arbitrary functions η^ϱ and η^ϱ_0, which are independent of x'^0 but depend on the other four coordinates x'^m, and which satisfy the conditions

$$\eta^\varrho = (\xi^\varrho)_0, \qquad \eta^\varrho_0 = \left(\frac{\partial \xi^\varrho}{\partial x'^0}\right)_0,$$

$(\quad)_0$ indicating the values at our special value of x'^0. Keeping this value of x'^0 fixed, it is now possible to calculate from (12) and (13) the values of the five quantities $\left(\dfrac{\partial^2 \xi^\sigma}{\partial x'^{02}}\right)_0$ as functions of x'^m, η^ϱ, η^ϱ_0, $\dfrac{\partial \eta^\varrho}{\partial x'^m}$, and $\dfrac{\partial \eta^\varrho_0}{\partial x'^m}$. Forming the higher derivatives $\left(\dfrac{\partial^n \xi^\sigma}{\partial x'^{0n}}\right)_0$ by successive differentiations, we get the MAC LAURIN series for ξ^ϱ,

$$\xi^\varrho(x'^\nu) = (\xi^\varrho)_0 + \left(\frac{\partial \xi^\varrho}{\partial x'^0}\right)_0 [x'^0 - (x'^0)_0] + \left(\frac{\partial^2 \xi^\sigma}{\partial x'^{02}}\right)_0 \frac{[x'^0 - (x'^0)_0]^2}{2!} + \cdots$$

This series converges under very wide assumptions as to the character of the actual functions.[1] If this value of $\xi^\varrho(x'^\nu)$ is used in (12) we get a coordinate system in which the conditions (11) are fullfilled.

The consequence of these conditions is that the general five-dimensional group of coordinate transformations otherwise permitted is now restricted by the demand that such a transformation must not carry over to a coordinate system, in which (11) are not satisfied. The restricted group of transformations is

$$\begin{cases} x^0 = x'^0 + \xi^0(x'^m) \\ x^k = \xi^k(x'^m) \end{cases} \tag{14}$$

This is just the under-group of five-dimensional transformations, characteristic of that form of the theory in which all quantities are taken to be independent of the fifth coordinate x^0. The use of the under-group instead of the general group is now the consequence of a coordinate choice, and imports no restriction on the general physical theory.

---

[1] GOURSAT, Cours d'Analyse, II, no. 385, 456. The coordinate transformation performed does not destroy the periodic character of the tensors.

§ 3. Transformation properties

The connection between the components of the fundamental tensors in five and in four dimensions in those coordinate systems which are now allowed is the same as that previously given by Klein [1].

$$\begin{aligned}
\gamma_{00} &= \alpha \text{ is an invariant,} \\
\gamma_{0m} &= \alpha\beta A_m, \\
\gamma_{mn} &= g_{mn} + \alpha\beta^2 A_m A_n, \\
\gamma^{00} &= \frac{1}{\alpha} + \beta^2 g^{mn} A_m A_n, \\
\gamma^{0m} &= -\beta g^{mn} A_n, \\
\gamma^{mn} &= g^{mn}.
\end{aligned} \tag{15}$$

The determinants γ and g are connected by the relation

$$\gamma = \alpha \cdot g, \tag{16}$$

as is easily verified (cf. [1], p. 198).

In all these formulas, the g_{mn} will now be periodic functions of x^0,

$$g_{mn} = \sum_{Z=-\infty}^{+\infty} g_{mn}^{(Z)} e^{2\pi i Z \cdot \frac{x^0}{l}}, \tag{17}$$

$g_{mn}^{(Z)}$ describing fields of the electric charge Ze. For $Z = 1$, the momentum canonically conjugate to x^0 ought to be

$$p_0 = \frac{e}{\beta c},$$

and this gives for the fundamental length l appearing in (17) the value

$$l = \frac{h\beta c}{e} \sim 10^{-30} \text{ cm.} \tag{18}$$

α and A_m are independent of x^0 and are functions of the space-time coordinates alone, as follows from (11).

If $P^{\mu\nu}$ are the fifteen components of a symmetric five-tensor, the ten components P^{mn} transform as the contravariant components of a symmetric four-tensor under a coordinate transformation pertaining to the group (14), the four components P_0^m behave as the contravariant components of a four-vector, and the fifteenth component P_{00} is an invariant.[1]

---

[1] An example of these rules is found in (15).

If Q^{rs} are the ten contravariant components of a symmetric four-tensor, defined by

$$Q^{rs} = P^{rs},$$

then we have for the covariant components,

$$Q_{rs} = P_{rs} - \beta A_r P_{0s} - \beta A_s P_{r0} + \beta^2 A_r A_s P_{00}, \tag{19}$$

as is easily verified.[1] The invariant P, given by (8), becomes then

$$P = \gamma^{\varrho\sigma} P_{\varrho\sigma} = \gamma^{rs} P_{rs} + \gamma^{r0} P_{r0} + \gamma^{0s} P_{0s} + \gamma^{00} P_{00}, \tag{20}$$

or, by inserting (15),

$$P = g^{rs} Q_{rs} + \frac{1}{\alpha} P_{00} = g_{rs} Q^{rs} + \frac{1}{\alpha} P_{00} = g_{rs} P^{rs} + \frac{1}{\alpha} P_{00}. \tag{21}$$

§ 4. A collection of rules and formulas

In the following calculations there will frequently appear expressions, built up in a way corresponding to (19), and resulting from a separation of indices as in (20). We will therefore introduce a new type of three-index symbols, namely

$$\tilde{\Gamma}_{r,0s} = \Gamma_{r,0s} - \beta A_r \Gamma_{0,0s} - \beta A_s \Gamma_{r,00} + \beta^2 A_r A_s \Gamma_{0,00},$$

$$\tilde{\Gamma}_{0,rs} = \Gamma_{0,rs} - \beta A_r \Gamma_{0,0s} - \beta A_s \Gamma_{0,r0} + \beta^2 A_r A_s \Gamma_{0,00},$$

and

$$\begin{aligned}\tilde{\Gamma}_{m,rs} = \Gamma_{m,rs} &- \beta A_m \Gamma_{0,rs} - \beta A_r \Gamma_{m,0s} - \beta A_s \Gamma_{m,r0} + \\ &+ \beta^2 A_m A_r \Gamma_{0,0s} + \beta^2 A_m A_s \Gamma_{0,r0} + \beta^2 A_r A_s \Gamma_{m,00} - \beta^3 A_m A_r A_s \Gamma_{0,00}.\end{aligned}$$

Inserting the values of the CHRISTOFFEL symbols from (2) and (15),

$$\Gamma_{0,00} = 0, \qquad \Gamma_{0,0s} = \frac{1}{2}\frac{\partial \alpha}{\partial x^s}, \qquad \Gamma_{r,00} = -\frac{1}{2}\frac{\partial \alpha}{\partial x^r}, \tag{22}$$

together with the corresponding expressions for $\Gamma_{r,0s}$, $\Gamma_{0,rs}$, and $\Gamma_{m,rs}$, we get by direct calculation,

$$\tilde{\Gamma}_{r,0s} = \frac{1}{2}\alpha\beta\left(\frac{\partial A_r}{\partial x^s} - \frac{\partial A_s}{\partial x^r}\right) + \frac{1}{2}\frac{\partial g_{rs}}{\partial x^0} = \frac{1}{2}\alpha\beta F_{sr} + \frac{1}{2}\frac{\partial g_{rs}}{\partial x^0}, \tag{23}$$

where the antisymmetric electro-magnetic field tensor F_{sr} has been introduced by the usual definition. Further,

$$\tilde{\Gamma}_{0,rs} = \frac{1}{2}\alpha\beta\left(\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r}\right) - \frac{1}{2}\frac{\partial g_{rs}}{\partial x^0}, \tag{24}$$

---

[1] KLEIN [1], p. 197.

and, finally,

$$\tilde{\Gamma}_{m,rs} = \tfrac{1}{2}(\partial_r g_{ms} + \partial_s g_{mr} - \partial_m g_{rs}) = \tilde{G}_{m,rs}.^{1} \tag{25}$$

Moreover, as well-known,

$$d g^{rs} = -g^{rm} g^{sn} d g_{mn}, \tag{26}$$

and

$$d \log \sqrt{g} = \frac{1}{\sqrt{g}} d\sqrt{g} = \frac{1}{2g} dg = \frac{1}{2} g^{rs} d g_{rs} = -\frac{1}{2} g_{rs} d g^{rs}. \tag{27}$$

Using, further, the fact that the determinants γ and g are connected by the relation $\gamma = \alpha \cdot g$, and that α is independent of x^0, we have

$$\frac{\partial}{\partial x^0} \log \sqrt{\gamma} = \frac{\partial}{\partial x^0} \log \sqrt{g}, \tag{28}$$

and

$$d_r \log \sqrt{\gamma} = \partial_r \log \sqrt{g} + \partial_r \log \sqrt{\alpha} = \partial_r \log \sqrt{g} + \frac{1}{\sqrt{\alpha}} \frac{\partial \sqrt{\alpha}}{\partial x^r}, \tag{29}$$

∂_r operating on $\sqrt{\alpha}$ being the ordinary derivative.

§ 5. The invariant P_{00}

After these preliminaries we will now calculate the components of the five-dimensional curvature tensor, and it follows from the transformation properties previously mentioned, that we get the closest correspondance to known four-dimensional expressions, if we choose the components P^{mn}, P^{m}_{0}, and P_{00}.

For the invariant P_{00} we get from (10),

$$P_{00} = \Gamma^{\varrho}_{0\sigma} \Gamma^{\sigma}_{0\varrho} + \frac{\partial^2}{\partial x^{0\,2}} \log \sqrt{\gamma} - \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^{\varrho}} (\sqrt{\gamma}\, \Gamma^{\varrho}_{00}). \tag{30}$$

By use of (22) and (23), the first term becomes, by separation of indices,

$$\begin{aligned}
\Gamma^{\varrho}_{0\sigma} \Gamma^{\sigma}_{0\varrho} &= \gamma^{\varrho\lambda} \gamma^{\sigma\tau} \Gamma_{\lambda,0\sigma} \Gamma_{\tau,0\varrho} = \\
&= g^{rl} g^{st} \tilde{\Gamma}_{l,0s} \tilde{\Gamma}_{t,0r} + \frac{1}{\alpha} g^{st} \Gamma_{0,0s} \Gamma_{t,00} + \frac{1}{\alpha} g^{rl} \Gamma_{l,00} \Gamma_{0,0r} = \\
&= \frac{1}{4} g^{rl} g^{st} \left(\alpha\beta F_{sl} + \frac{\partial g_{sl}}{\partial x^0}\right) \left(\alpha\beta F_{rt} + \frac{\partial g_{rt}}{\partial x^0}\right) - \frac{1}{2\alpha} g^{st} \frac{\partial \alpha}{\partial x^s} \frac{\partial \alpha}{\partial x^t},
\end{aligned}$$

---

[1] Cf. (2 c). It should be noticed that, as required by the fundamental gauge invariance of the theory, the operators ∂_r operating on a term in the FOURIER expansion (17), which corresponds to particles of charge Ze, go over into (by means of (18)),

$$\partial_r = \frac{\partial}{\partial x^r} - \beta A_r \cdot \frac{2\pi i Z}{l} = \frac{\partial}{\partial x^r} - \frac{iZe}{\hbar c} A_r,$$

which are the usual operators appearing in any gauge-invariant theory of electric particles.

or, by changing some indices and using the relation (26),

$$\Gamma^{\varrho}_{0\sigma}\,\Gamma^{\sigma}_{0\varrho} = -\frac{1}{4}\,\alpha^2\,\beta^2\,F_{rs}\,F^{rs} - \frac{1}{4}\,\frac{\partial g^{rs}}{\partial x^0}\,\frac{\partial g_{rs}}{\partial x^0} - 2\,g^{rs}\,\frac{\partial\sqrt{\alpha}}{\partial x^r}\,\frac{\partial\sqrt{\alpha}}{\partial x^s}, \tag{31}$$

the other terms disappearing by reasons of symmetry.

The second term in (30) becomes, by means of (28) and (27),

$$\frac{\partial^2}{\partial x^{0\,2}}\log\sqrt{\gamma} = \frac{\partial}{\partial x^0}\left(\frac{1}{2}\,g^{rs}\,\frac{\partial g_{rs}}{\partial x^0}\right) = \frac{1}{2}\,g^{rs}\,\frac{\partial^2 g_{rs}}{\partial x^{0^2}} + \frac{1}{2}\,\frac{\partial g^{rs}}{\partial x^0}\,\frac{\partial g_{rs}}{\partial x^0}, \tag{32}$$

and the third term

$$\begin{aligned}
-\frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^{\varrho}}\left(\sqrt{\gamma}\,\gamma^{\varrho\sigma}\,\Gamma_{\sigma,00}\right) &= -\frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^r}\left(\sqrt{\gamma}\,g^{rs}\,\Gamma_{s,00}\right) - \\
&\quad - \frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^0}\left(\sqrt{\gamma}\cdot\beta\,g^{rs}\,A_r\,\Gamma_{s,00}\right) = \\
&= -\frac{1}{\sqrt{\gamma}}\,\partial_r\left(\sqrt{\gamma}\,g^{rs}\,\Gamma_{s,00}\right),
\end{aligned}$$

since A_r is independent of x^0. Inserting

$$\Gamma_{s,00} = -\frac{1}{2}\,\frac{\partial\alpha}{\partial x^s} = -\sqrt{\alpha}\,\frac{\partial\sqrt{\alpha}}{\partial x^s},$$

and using (29) and (27), we get

$$-\frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^{\varrho}}\left(\sqrt{\gamma}\,\gamma^{\varrho\sigma}\,\Gamma_{\sigma,00}\right) = \sqrt{\alpha}\cdot\frac{1}{\sqrt{g}}\,\partial_r\left(\sqrt{g}\,g^{rs}\,\frac{\partial\sqrt{\alpha}}{\partial x^s}\right) + 2\,g^{rs}\,\frac{\partial\sqrt{\alpha}}{\partial x^r}\,\frac{\partial\sqrt{\alpha}}{\partial x^s}. \tag{33}$$

Adding together (31), (32), and (33)

$$P_{00} = -\frac{1}{4}\,\alpha^2\,\beta^2\,F_{rs}\,F^{rs} + \frac{1}{2}\,g^{rs}\,\frac{\partial^2 g_{rs}}{\partial x^{0\,2}} + \frac{1}{4}\,\frac{\partial g^{rs}}{\partial x^0}\,\frac{\partial g_{rs}}{\partial x^0} + \sqrt{\alpha}\cdot\tilde{\Box}\,\sqrt{\alpha}, \tag{34}$$

where $\tilde{\Box}$ is the covariant four-dimensional LAPLACE operator with the derivatives $\frac{\partial}{\partial x^r}$ replaced by the operators ∂_r,

$$\tilde{\Box}\,\sqrt{\alpha} = \frac{1}{\sqrt{g}}\,\partial_r\left(\sqrt{g}\,g^{rs}\,\frac{\partial\sqrt{\alpha}}{\partial x^s}\right) = g^{rs}\left(\frac{\partial^2\sqrt{\alpha}}{\partial x^r\,\partial x^s} - \tilde{G}^m_{rs}\,\frac{\partial\sqrt{\alpha}}{\partial x^m}\right).$$

§ 6. The four-vector $\underset{0}{P}^m$

From (10) we find

$$\underset{0}{P}^m = \gamma^{m\nu} P_{\nu 0} = \gamma^{m\nu} \Gamma^{\sigma}_{\nu\varrho} \Gamma^{\varrho}_{0\sigma} + \gamma^{m\nu} \frac{\partial^2}{\partial x^\nu \partial x^0} \log \sqrt{\gamma} -$$

$$- \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\varrho} (\sqrt{\gamma}\, \gamma^{m\nu} \Gamma^{\varrho}_{\nu 0}) + \Gamma^{\varrho}_{\nu 0} \frac{\partial \gamma^{m\nu}}{\partial x^\varrho}. \tag{35}$$

We take together the first and the last term and use (26),

$$\gamma^{m\nu} \Gamma^{\sigma}_{\nu\varrho} \Gamma^{\varrho}_{0\sigma} + \Gamma^{\varrho}_{\nu 0} \frac{\partial \gamma^{m\nu}}{\partial x^\varrho} = \gamma^{m\nu} \Gamma^{\sigma}_{\nu\varrho} \Gamma^{\varrho}_{0\sigma} + \Gamma^{\varrho}_{0\sigma} \frac{\partial \gamma^{m\sigma}}{\partial x^\varrho} =$$

$$= \gamma^{m\nu} \gamma^{\varrho\lambda} \gamma^{\sigma\tau} \Gamma_{\lambda, 0\sigma} \left(\Gamma_{\tau, \nu\varrho} - \frac{\partial \gamma_{\nu\tau}}{\partial x^\varrho} \right) = - \gamma^{m\nu} \gamma^{\varrho\lambda} \gamma^{\sigma\tau} \Gamma_{\lambda, 0\sigma} \Gamma_{\nu, \varrho\tau} =$$

$$= - g^{mn} \left(g^{rl} g^{st} \tilde{\Gamma}_{l, 0s} \tilde{\Gamma}_{n, rt} + \frac{1}{\alpha} g^{st} \Gamma_{0, 0s} \tilde{\Gamma}_{n, 0t} + \frac{1}{\alpha} g^{rl} \Gamma_{l, 00} \tilde{\Gamma}_{n, r0} \right).$$

The last two terms in the parenthesis cancel each other, and in the first term only the symmetric part of $\tilde{\Gamma}_{l, 0s}$ will survive, since $g^{rl} g^{st} \tilde{\Gamma}_{n, rt}$ is symmetric in l and s. Thus, by means of (25), (23), and (26),

$$\gamma^{m\nu} \Gamma^{\sigma}_{\nu\varrho} \Gamma^{\varrho}_{0\sigma} + \Gamma^{\varrho}_{\nu 0} \frac{\partial \gamma^{m\nu}}{\partial x^\varrho} = \frac{1}{2} \frac{\partial g^{rs}}{\partial x^0} \tilde{G}^{m}_{rs}. \tag{36}$$

The second term of (35) is,

$$\gamma^{m\nu} \frac{\partial^2}{\partial x^\nu \partial x^0} \log \sqrt{\gamma} = g^{mn} \partial_n \frac{\partial}{\partial x^0} \log \sqrt{g}. \tag{37}$$

and the third term

$$- \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\varrho} (\sqrt{\gamma}\, \gamma^{m\nu} \gamma^{\varrho\lambda} \Gamma_{\lambda, \nu 0}) = - \frac{1}{\sqrt{\gamma}} \partial_r (\sqrt{\gamma}\, g^{mn} g^{rl} \tilde{\Gamma}_{l, n0}) -$$

$$- \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^0} \left(\sqrt{\gamma}\, g^{mn} \cdot \frac{1}{\alpha} \Gamma_{0, n0} \right) =$$

$$= - \frac{1}{2\sqrt{\gamma}} \partial_r \left(\sqrt{\gamma}\, \alpha \beta F^{mr} - \sqrt{\gamma} \frac{\partial g^{mr}}{\partial x^0} \right) - \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^0} \left(\sqrt{\gamma}\, g^{mn} \cdot \frac{1}{\sqrt{\alpha}} \frac{\partial \sqrt{\alpha}}{\partial x^n} \right) -$$

$$= - \frac{1}{2\sqrt{\alpha \cdot g}} \partial_r (\sqrt{g}\, \alpha^{3/2} \beta F^{mr}) + \frac{1}{2\sqrt{g}} \partial_r \left(\sqrt{g} \frac{\partial g^{mr}}{\partial x^0} \right) + \frac{1}{2\sqrt{\alpha}} \frac{\partial \sqrt{\alpha}}{\partial x^r} \frac{\partial g^{mr}}{\partial x^0} -$$

$$- \frac{1}{\sqrt{\alpha}} \frac{\partial \sqrt{\alpha}}{\partial x^n} \cdot \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} (\sqrt{g}\, g^{mn}) =$$

$$= -\frac{1}{2\sqrt{a\cdot g}}\,\partial_n\left(\sqrt{g}\,a^{3/2}\beta\,F^{mn}\right) + \frac{1}{2\sqrt{g}}\,\partial_n\left(\sqrt{g}\,\frac{\partial g^{mn}}{\partial x^0}\right) -$$

$$- \frac{1}{\sqrt{a}}\,\frac{\partial\sqrt{a}}{\partial x^n}\left(g^{mn}\cdot\frac{1}{\sqrt{g}}\,\frac{\partial\sqrt{g}}{\partial x^0} + \frac{1}{2}\,\frac{\partial g^{mn}}{\partial x^0}\right) \qquad (38)$$

Adding together (36), (37), and (38),

$$P^m_0 = -\frac{1}{2}\beta\cdot\frac{1}{\sqrt{a\cdot g}}\,\partial_n\left(\sqrt{g}\,a^{3/2}F^{mn}\right) + g^{mn}\,\partial_n\frac{\partial}{\partial x^0}\log\sqrt{g} + \frac{1}{2\sqrt{g}}\,\partial_n\left(\sqrt{g}\,\frac{\partial g^{mn}}{\partial x^0}\right) +$$

$$+ \frac{1}{2}\,\frac{\partial g^{rs}}{\partial x^0}\,\tilde{G}^m_{rs} - \frac{1}{\sqrt{a}}\,\frac{\partial\sqrt{a}}{\partial x^n}\cdot\frac{1}{2g}\,\frac{\partial}{\partial x^0}\left(g\,g^{mn}\right). \qquad (39)$$

§ 7. The four-tensor P^{mn}

In the expression

$$P^{mn} = \gamma^{m\varrho}\gamma^{n\sigma}\,\Gamma^{\lambda}_{\varrho\tau}\,\Gamma^{\tau}_{\sigma\lambda} + \gamma^{m\varrho}\gamma^{n\sigma}\,\frac{\partial^2}{\partial x^{\varrho}\,\partial x^{\sigma}}\log\sqrt{\gamma} - \frac{1}{\sqrt{\gamma}}\,\gamma^{m\varrho}\gamma^{n\sigma}\,\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{\gamma}\,\Gamma^{\lambda}_{\varrho\sigma}\right) \qquad (40)$$

we will calculate the three terms separately.

$$\gamma^{m\varrho}\gamma^{n\sigma}\gamma^{\lambda\iota}\gamma^{\tau\varkappa}\,\Gamma_{\iota,\varrho\tau}\,\Gamma_{\varkappa,\sigma\lambda} =$$

$$= g^{mr}g^{ns}\left[g^{li}g^{tk}\,\tilde{\Gamma}_{i,rt}\,\tilde{\Gamma}_{k,sl} + \frac{1}{a}\,g^{tk}\,\tilde{\Gamma}_{0,rt}\,\tilde{\Gamma}_{k,s0} + \frac{1}{a}\,g^{li}\,\tilde{\Gamma}_{i,r0}\,\tilde{\Gamma}_{0,sl} + \frac{1}{a^2}\,\Gamma_{0,r0}\,\Gamma_{0,s0}\right] =$$

$$= g^{mr}g^{ns}\left[\tilde{G}^l_{rt}\,\tilde{G}^t_{sl} + \frac{1}{a}\,g^{lt}\left(\tilde{\Gamma}_{0,rt}\,\tilde{\Gamma}_{l,s0} + \tilde{\Gamma}_{t,r0}\,\tilde{\Gamma}_{0,sl}\right) + \frac{1}{4a^2}\,\frac{\partial a}{\partial x^r}\,\frac{\partial a}{\partial x^s}\right]. \qquad (41)$$

The expression will be left in this form for the present.

In the second, as well as in the third, term of (40) we must be careful, since there will appear derivatives of the electro-magnetic potentials A_s, when γ^{n0} is replaced by $-\beta g^{ns}A_s$,

$$\gamma^{m\varrho}\gamma^{n\sigma}\,\frac{\partial^2}{\partial x^{\varrho}\,\partial x^{\sigma}}\log\sqrt{\gamma} =$$

$$= \gamma^{m\varrho}\,\frac{\partial}{\partial x^{\varrho}}\left(\gamma^{n\sigma}\,\frac{\partial}{\partial x^{\sigma}}\log\sqrt{\gamma}\right) - \gamma^{m\varrho}\,\frac{\partial\gamma^{n\sigma}}{\partial x^{\varrho}}\,\frac{\partial}{\partial x^{\sigma}}\log\sqrt{\gamma} =$$

$$= g^{mr}\,\partial_r\left(g^{ns}\,\partial_s\log\sqrt{\gamma}\right) - g^{mr}\,\partial_r\,g^{ns}\cdot\partial_s\log\sqrt{\gamma} + \beta\,g^{mr}g^{ns}\,\frac{\partial A_s}{\partial x^r}\,\frac{\partial}{\partial x^0}\log\sqrt{\gamma}$$

$$= g^{mr}g^{ns}\left(\partial_r\partial_s\log\sqrt{\gamma} + \beta\,\frac{\partial A_s}{\partial x^r}\,\frac{\partial}{\partial x^0}\log\sqrt{\gamma}\right).$$

This expression ought to be written in a form symmetric in m and n, since the operators ∂_r and ∂_s are not interchangeable. Thus, using (29),

$$\gamma^{m\varrho}\gamma^{n\sigma}\frac{\partial^2}{\partial x^\varrho\,\partial x^\sigma}\log\sqrt{\gamma} =$$

$$= g^{mr}g^{ns}\left[\frac{1}{2}(\partial_r\partial_s+\partial_s\partial_r)\log\sqrt{g}+\frac{\partial}{\partial x^r}\left(\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}}{\partial x^s}\right)+\right.$$

$$\left.+\frac{1}{2}\beta\left(\frac{\partial A_s}{\partial x^r}+\frac{\partial A_r}{\partial x^s}\right)\frac{\partial}{\partial x^0}\log\sqrt{g}\right]. \qquad (42)$$

For the last term of (40) we get

$$-\gamma^{m\varrho}\gamma^{n\sigma}\cdot\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\lambda}(\sqrt{\gamma}\,\gamma^{\lambda\tau}\,\Gamma_{\tau,\varrho\sigma}) =$$

$$= -g^{mr}g^{ns}\left[\frac{1}{\sqrt{\gamma}}\partial_l(\sqrt{\gamma}\,g^{lt}\,\tilde{\Gamma}_{t,rs})+\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^0}\left(\sqrt{\gamma}\cdot\frac{1}{a}\tilde{\Gamma}_{0,rs}\right)+\right.$$

$$\left.+\beta g^{lt}\,\tilde{\Gamma}_{t,r0}\frac{\partial A_s}{\partial x^l}+\beta g^{lt}\,\tilde{\Gamma}_{t,0s}\frac{\partial A_r}{\partial x^l}\right] =$$

$$= -g^{mr}g^{ns}\left[\frac{1}{\sqrt{g}}\partial_l(\sqrt{g}\,\tilde{G}^l_{rs})+\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}}{\partial x^l}\tilde{G}^l_{rs}+\right.$$

$$+\frac{1}{2a}\cdot a\beta\left(\frac{\partial A_r}{\partial x^s}+\frac{\partial A_s}{\partial x^r}\right)\cdot\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^0}-$$

$$\left.-\frac{1}{2a}\cdot\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g_{rs}}{\partial x^0}\right)+\beta g^{lt}\left(\tilde{\Gamma}_{t,r0}\frac{\partial A_s}{\partial x^l}+\tilde{\Gamma}_{l,s0}\frac{\partial A_r}{\partial x^t}\right)\right], \qquad (43)$$

where we have inserted the value of $\tilde{\Gamma}_{0,rs}$ from (24). Adding together (41), (42), and (43), we see, that the sum of the first term in each of these expressions will become

$$g^{mr}g^{ns}\left[\tilde{G}^l_{rt}\tilde{G}^t_{sl}+\frac{1}{2}(\partial_r\partial_s+\partial_s\partial_r)\log\sqrt{g}-\frac{1}{\sqrt{g}}\partial_l(\sqrt{g}\,\tilde{G}^l_{rs})\right] = \tilde{R}^{mn},$$

where $\tilde{R}^{mn}$ is the curvature tensor (9) with all derivatives $\dfrac{\partial}{\partial x^r}$ replaced by ∂_r. Further, we see, that the terms containing

$$\frac{\partial A_r}{\partial x^s}+\frac{\partial A_s}{\partial x^r}$$

in (42) and (43), cancel each other, and that the terms containing $\tilde{\Gamma}_{t,r0}$ in (41) and (43) give, by using (26) together with (23) and (24),

$$g^{mr}g^{ns}\cdot\frac{1}{\alpha}g^{lt}\tilde{\Gamma}_{t,r0}\left(\tilde{\Gamma}_{0,sl}-\alpha\beta\frac{\partial A_s}{\partial x^l}\right)=$$

$$=g^{mr}g^{ns}\cdot\frac{1}{\alpha}g^{lt}\cdot\frac{1}{4}\left(\alpha\beta F_{rt}+\frac{\partial g_{rt}}{\partial x^0}\right)\left(\alpha\beta F_{sl}-\frac{\partial g_{sl}}{\partial x^0}\right)=$$

$$=\frac{1}{4}\alpha\beta^2F^m{}_tF^{nt}-\frac{1}{4\alpha}g_{rs}\frac{\partial g^{mr}}{\partial x^0}\frac{\partial g^{ns}}{\partial x^0}+\text{cross terms},$$

the cross terms disappearing, when we add the corresponding expression for the terms containing $\tilde{\Gamma}_{l,s0}$.

The result is,

$$P^{mn}=\tilde{R}^{mn}+\frac{1}{2}\alpha\beta^2F^m{}_rF^{nr}-\frac{1}{2\alpha}g_{rs}\frac{\partial g^{mr}}{\partial x^0}\frac{\partial g^{ns}}{\partial x^0}+$$

$$+\frac{1}{2\alpha}g^{mr}g^{ns}\cdot\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g_{rs}}{\partial x^0}\right)+$$

$$+g^{mr}g^{ns}\left[\frac{1}{4\alpha^2}\frac{\partial\alpha}{\partial x^r}\frac{\partial\alpha}{\partial x^s}+\frac{\partial}{\partial x^r}\left(\frac{1}{\sqrt{\alpha}}\frac{\partial\sqrt{\alpha}}{\partial x^s}\right)-\frac{1}{\sqrt{\alpha}}\frac{\partial\sqrt{\alpha}}{\partial x^l}\tilde{G}^l_{rs}\right]. \tag{44}$$

This expression can be written in a more condensed form, since

$$g^{mr}g^{ns}\cdot\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g_{rs}}{\partial x^0}\right)=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\,g^{mr}g^{ns}\frac{\partial g_{rs}}{\partial x^0}\right)-\frac{\partial g_{rs}}{\partial x^0}\frac{\partial}{\partial x^0}(g^{mr}g^{ns})=$$

$$=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g^{mn}}{\partial x^0}\right)+g_{rl}g_{st}\frac{\partial g^{lt}}{\partial x^0}\left(g^{mr}\frac{\partial g^{ns}}{\partial x^0}+g^{ns}\frac{\partial g^{mr}}{\partial x^0}\right)=$$

$$=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g^{mn}}{\partial x^0}\right)+2g_{rs}\frac{\partial g^{mr}}{\partial x^0}\frac{\partial g^{ns}}{\partial x^0},$$

where the indices have been changed in the last term. Further,

$$\frac{1}{4\alpha^2}\frac{\partial\alpha}{\partial x^r}\frac{\partial\alpha}{\partial x^s}=\frac{1}{\alpha}\frac{\partial\sqrt{\alpha}}{\partial x^r}\frac{\partial\sqrt{\alpha}}{\partial x^s},$$

and

$$\frac{\partial}{\partial x^r}\left(\frac{1}{\sqrt{\alpha}}\frac{\partial\sqrt{\alpha}}{\partial x^s}\right)=\frac{1}{\sqrt{\alpha}}\frac{\partial^2\sqrt{\alpha}}{\partial x^r\partial x^s}-\frac{1}{\alpha}\frac{\partial\sqrt{\alpha}}{\partial x^r}\frac{\partial\sqrt{\alpha}}{\partial x^s}.$$

By means of these relations, (42) goes over into

$$P^{mn}=\tilde{R}^{mn}+\frac{1}{2}\alpha\beta^2F^m{}_rF^{nr}-\frac{1}{2\alpha}\cdot\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g^{mn}}{\partial x^0}\right)+\frac{1}{2\alpha}g_{rs}\frac{\partial g^{mr}}{\partial x^0}\frac{\partial g^{ns}}{\partial x^0}+$$

$$+g^{mr}g^{ns}\cdot\frac{1}{\sqrt{\alpha}}\left(\frac{\partial^2\sqrt{\alpha}}{\partial x^r\partial x^s}-\frac{\partial\sqrt{\alpha}}{\partial x^l}\tilde{G}^l_{rs}\right). \tag{45}$$

The last parenthesis represents the covariant second derivative of $\sqrt{\alpha}$.

§ 8. **The tensor $\Gamma_{\mu\nu}$.**

We are now going to write down the expressions for the components of the tensor $\Gamma_{\mu\nu}$ entering into the field equations (6),

$$\Gamma_{\mu\nu} = P_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} P = -\frac{\beta^2}{2}\Theta_{\mu\nu}, \tag{6}$$

where P is given by (21),

$$P = \gamma^{\varrho\sigma} P_{\varrho\sigma} = g_{rs} P^{rs} + \frac{1}{\alpha} P_{00}. \tag{21}$$

From (34) we have

$$\frac{1}{\alpha} P_{00} = -\frac{1}{4}\alpha\beta^2 F_{rs}F^{rs} + \frac{1}{4\alpha}\frac{\partial g^{rs}}{\partial x^0}\frac{\partial g_{rs}}{\partial x^0} + \frac{1}{2\alpha} g^{rs}\frac{\partial^2 g_{rs}}{\partial x^{0\,2}} + \frac{1}{\sqrt{\alpha}}\tilde{\Box}\sqrt{\alpha},$$

and from (44), after changing some indices[1],

$$g_{mn}P^{mn} = \tilde{R} + \frac{1}{2}\alpha\beta^2 F_{rs}F^{rs} + \frac{1}{2\alpha}\frac{\partial g^{rs}}{\partial x^0}\frac{\partial g_{rs}}{\partial x^0} + \\ + \frac{1}{2\alpha} g^{rs}\cdot\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left(\sqrt{g}\frac{\partial g_{rs}}{\partial x^0}\right) + \frac{1}{\sqrt{\alpha}}\tilde{\Box}\sqrt{\alpha}.$$

Inserting these expressions into (21), we get

$$P = \tilde{R} + \frac{1}{4}\alpha\beta^2 F_{rs}F^{rs} + \frac{3}{4\alpha}\frac{\partial g^{rs}}{\partial x^0}\frac{\partial g_{rs}}{\partial x^0} + \frac{1}{\alpha} g^{rs}\frac{\partial^2 g_{rs}}{\partial x^{0\,2}} + \\ + \frac{1}{2\alpha} g^{rs}\frac{\partial g_{rs}}{\partial x^0}\cdot\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^0} + \frac{2}{\sqrt{\alpha}}\tilde{\Box}\sqrt{\alpha}. \tag{46}$$

Further, from (6) and (27),

$$\Gamma_{00} = P_{00} - \frac{1}{2}\alpha P = -\frac{1}{2}\alpha\tilde{R} - \frac{3}{8}\alpha^2\beta^2 F_{rs}F^{rs} - \\ - \frac{1}{8}\frac{\partial g^{rs}}{\partial x^0}\frac{\partial g_{rs}}{\partial x^0} - \frac{1}{2}\left(\frac{\partial}{\partial x^0}\log\sqrt{g}\right)^2. \tag{47}$$

It is of special interest to notice that the derivatives of α have altogether disappeared here, as well as the second derivatives of g_{rs} with respect to x^0.

For the mixed components we have, since $\gamma_0^m = 0$,

$$\Gamma_0^m = P_0^m - \tfrac{1}{2}\gamma_0^m P = P_0^m,$$

and for the contravariant components,

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[1] Cf. also (45).

$$\Gamma^{mn} = \tilde{G}^{mn} + \frac{1}{2}\alpha\beta^2 S^{mn} + \frac{1}{2\alpha} g_{rs} \frac{\partial g^{mr}}{\partial x^0} \frac{\partial g^{ns}}{\partial x^0} - \frac{3}{8\alpha} g^{mn} \frac{\partial g^{rs}}{\partial g^0} \frac{\partial g_{rs}}{\partial x^0} -$$
$$- \frac{1}{2\alpha} g^{mn} g^{rs} \frac{\partial^2 g_{rs}}{\partial x^{0\,2}} - \frac{1}{2\alpha} \cdot \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} \left(\sqrt{g} \frac{\partial g^{mn}}{\partial x^0} \right) - \frac{1}{2\alpha} g^{mn} \left(\frac{\partial}{\partial x^0} \log \sqrt{g} \right)^2 +$$
$$+ g^{mr} g^{ns} \cdot \frac{1}{\sqrt{\alpha}} \left(\frac{\partial^2 \sqrt{\alpha}}{\partial x^r \partial x^s} - \frac{\partial \sqrt{\alpha}}{\partial x^l} \tilde{G}^l_{rs} \right) - \frac{1}{\sqrt{\alpha}} g^{mn} \tilde{\Box} \sqrt{\alpha}, \tag{48}$$

where we have introduced the electro-magnetic energy-momentum tensor,

$$S^{mn} = F^m{}_r F^{nr} - \tfrac{1}{4} g^{mn} F_{rs} F^{rs},$$

together with the tensor $\tilde{G}^{mn}$, defined by

$$\tilde{G}^{mn} = \tilde{R}^{mn} - \tfrac{1}{2} g^{mn} \tilde{R}.$$

§ 9. The Lagrangian Λ

It may be convenient, at this place, to calculate also the Lagrangian of the five-dimensional field, although the value of this quantity will not be explicitly used in this paper. The Lagrangian is given by (5),

$$\Lambda = \gamma^{\mu\nu} (\Gamma^{\varrho}_{\mu\sigma} \Gamma^{\sigma}_{\nu\varrho} - \Gamma^{\varrho}_{\mu\nu} \Gamma^{\sigma}_{\sigma\varrho}).$$

We get immediately

$$\Lambda = g^{mn} g^{rl} g^{st} (\tilde{\Gamma}_{l,ms} \tilde{\Gamma}_{t,nr} - \tilde{\Gamma}_{l,mn} \tilde{\Gamma}_{t,sr}) +$$
$$+ \frac{1}{\alpha} g^{rl} g^{st} (\tilde{\Gamma}_{l,0s} \tilde{\Gamma}_{t,0r} - \Gamma_{l,00} \tilde{\Gamma}_{t,sr}) + \frac{1}{\alpha} g^{mn} g^{st} (\tilde{\Gamma}_{0,ms} \tilde{\Gamma}_{t,n0} - \tilde{\Gamma}_{0,mn} \tilde{\Gamma}_{t,s0}) +$$
$$+ \frac{1}{\alpha} g^{mn} g^{rl} (\tilde{\Gamma}_{l,m0} \tilde{\Gamma}_{0,nr} - \tilde{\Gamma}_{l,mn} \Gamma_{0,0r}) + \frac{1}{\alpha^2} g^{mn} \Gamma_{0,m0} \Gamma_{0,n0} + \frac{1}{\alpha^2} g^{st} \Gamma_{0,0s} \Gamma_{t,00}.$$

The last two terms cancel each other. Changing some indices,

$$\Lambda = g^{mn} (\tilde{G}^r_{ms} \tilde{G}^s_{nr} - \tilde{G}^r_{mn} \tilde{G}^s_{sr}) + \frac{1}{\alpha} g^{mn} g^{rs} (\tilde{\Gamma}_{m,0s} \tilde{\Gamma}_{r,0n} - \Gamma_{m,00} \tilde{\Gamma}_{r,sn} +$$
$$+ \tilde{\Gamma}_{0,ms} \tilde{\Gamma}_{r,n0} - \tilde{\Gamma}_{0,mn} \tilde{\Gamma}_{r,s0} + \tilde{\Gamma}_{s,m0} \tilde{\Gamma}_{0,nr} - \tilde{\Gamma}_{s,mn} \Gamma_{0,0r}).$$

Here we introduce the four-dimensional Lagrangian

$$\tilde{L} = g^{mn} (\tilde{G}^r_{ms} \tilde{G}^s_{nr} - \tilde{G}^r_{mn} \tilde{G}^s_{sr}),$$

and change some indices and the order of the terms[1],

---

[1] $g^{mn} g^{rs} \tilde{\Gamma}_{s,m0} \tilde{\Gamma}_{0,nr} = g^{mn} g^{rs} \tilde{\Gamma}_{r,n0} \tilde{\Gamma}_{0,ms}$, $\Gamma_{0,0r} = - \Gamma_{r,00}$.

$$\Lambda = \tilde{L} + \frac{1}{\alpha} g^{mn} g^{rs} [\tilde{\Gamma}_{r,0n} (\tilde{\Gamma}_{m,0s} + 2\tilde{\Gamma}_{0,ms}) - \tilde{\Gamma}_{0,mn} \tilde{\Gamma}_{r,s0} - \Gamma_{m,00} (\tilde{\Gamma}_{r,sn} - \tilde{\Gamma}_{n,rs})].$$

Inserting the values of the three-index symbols from (23), (24), and (25), and raising indices by means of (26),

$$\begin{aligned} \Lambda = \tilde{L} + \frac{1}{4\alpha} \left(\alpha\beta F^{ms} - \frac{\partial g^{ms}}{\partial x^0} \right) \cdot \\ \cdot \left[\alpha\beta F_{sm} + \frac{\partial g_{ms}}{\partial x^0} + 2\alpha\beta \left(\frac{\partial A_m}{\partial x^s} + \frac{\partial A_s}{\partial x^m} \right) - 2 \frac{\partial g_{ms}}{\partial x^0} \right] - \\ - \frac{1}{4\alpha} g^{mn} g^{rs} \left[\alpha\beta \left(\frac{\partial A_m}{\partial x^n} + \frac{\partial A_n}{\partial x^m} \right) - \frac{\partial g_{mn}}{\partial x^0} \right] \cdot \frac{\partial g_{rs}}{\partial x^0} + \\ + \frac{1}{2\alpha} \frac{\partial \alpha}{\partial x^m} (g^{mn} \tilde{G}^s_{sn} - g^{rs} \tilde{G}^m_{rs}), \end{aligned}$$

the anti-symmetric part of $\tilde{\Gamma}_{r,s0}$ disappearing because of the multiplication by g^{rs}. Using further properties of symmetry, we may reduce the expression to

$$\begin{aligned} \Lambda = \tilde{L} - \frac{1}{4} \alpha\beta^2 F_{rs} F^{rs} + \frac{1}{4\alpha} \frac{\partial g^{rs}}{\partial x^0} \frac{\partial g_{rs}}{\partial x^0} - \\ - \frac{1}{2} \beta \left(\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} \right) \left(\frac{\partial g^{rs}}{\partial x^0} + \frac{1}{2} g^{rs} g^{mn} \frac{\partial g_{mn}}{\partial x^0} \right) - \\ - \frac{1}{4\alpha} g^{mn} \frac{\partial g_{mn}}{\partial x^0} g^{rs} \frac{\partial g^{rs}}{\partial x^0} + \frac{1}{2\alpha} \frac{\partial \alpha}{\partial x^m} (g^{mn} \tilde{G}^s_{sn} - g^{rs} \tilde{G}^m_{rs}). \end{aligned}$$

By means of (27), this can be written

$$\begin{aligned} \Lambda = \tilde{L} - \frac{1}{4} \alpha\beta^2 F_{rs} F^{rs} + \frac{1}{4\alpha} \frac{\partial g^{rs}}{\partial x^0} \frac{\partial g_{rs}}{\partial x^0} - \\ - \frac{1}{2} \beta \left(\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r} \right) \cdot \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} (\sqrt{g}\, g^{rs}) - \\ - \frac{1}{\alpha} \left(\frac{\partial}{\partial x^0} \log \sqrt{g} \right)^2 + \frac{1}{\sqrt{\alpha}} \frac{\partial \sqrt{\alpha}}{\partial x^m} (g^{mn} \tilde{G}^s_{sn} - g^{rs} \tilde{G}^m_{rs}). \end{aligned} \qquad (49)$$

It must be remembered, that Λ is not a general invariant. When we form the invariant $\int \Lambda \sqrt{g}\, d x^0\, d x^1\, d x^2\, d x^3\, d x^4$, the term containing $\frac{\partial A_r}{\partial x^s} + \frac{\partial A_s}{\partial x^r}$ disappears by partial integration.

§ 10. Non-charged fields

In the case of non-charged fields, i. e. when the fundamental tensor is independent of x^0, so that in the FOURIER expansion (17) only the term $\overset{(0)}{g}_{mn}$ survives, the field equations are greatly simplified, since all derivatives with respect to x^0 disappear. In this case, the expressions (34), (39), (45), (47), (48), and (49) go over into[1]

$$P_{00} = -\frac{1}{4}\alpha^2\beta^2 F_{rs}F^{rs} + \sqrt{\alpha}\cdot\Box\sqrt{\alpha}, \tag{34 a}$$

$$P_0^m = -\frac{1}{2}\beta\cdot\frac{1}{\sqrt{\alpha\cdot g}}\frac{\partial}{\partial x^n}\left(\sqrt{g}\,\alpha^{3/2}F^{mn}\right), \tag{39 a}$$

$$P^{mn} = R^{mn} + \frac{1}{2}\alpha\beta^2 F^m{}_r F^{nr} + g^{mr}g^{ns}\cdot\frac{1}{\sqrt{\alpha}}\left(\frac{\partial^2\sqrt{\alpha}}{\partial x^r\,\partial x^s} - \frac{\partial\sqrt{\alpha}}{\partial x^l}G^l_{rs}\right), \tag{45 a}$$

$$\Gamma_{00} = -\frac{1}{2}\alpha R - \frac{3}{8}\alpha^2\beta^2 F_{rs}F^{rs}, \tag{47 a}$$

$$\begin{aligned}\Gamma^{mn} = G^{mn} + \frac{1}{2}\alpha\beta^2 S^{mn} &+ \\ &+ g^{mr}g^{ns}\cdot\frac{1}{\sqrt{\alpha}}\left(\frac{\partial^2\sqrt{\alpha}}{\partial x^r\,\partial x^s} - \frac{\partial\sqrt{\alpha}}{\partial x^l}G^l_{rs}\right) - \frac{1}{\sqrt{\alpha}}g^{mn}\Box\sqrt{\alpha}.\end{aligned} \tag{48 a}$$

$$\Lambda = L - \frac{1}{4}\alpha\beta^2 F_{rs}F^{rs} + \frac{1}{\sqrt{\alpha}}\frac{\partial\sqrt{\alpha}}{\partial x^m}\left(g^{mn}G^s_{sn} - g^{rs}G^m_{rs}\right). \tag{49 a}$$

§ 11. Central-symmetric static fields

We will now give the components of the curvature tensor $P_{\mu\nu}$ in the case of spherical symmetry. For the line element we write

$$d\sigma^2 = \gamma_{\mu\nu}\,dx^\mu\,dx^\nu = g_{mn}\,dx^m\,dx^n + \alpha\,(dx^0 + \beta A_m\,dx^m)^2.$$

As to the electro-magnetic potential vector we assume that only the component $A_4 = A$ is $\neq 0$ (this means that we disregard spin effects). The condition of spherical symmetry is introduced by the supposition

$$ds^2 = g_{mn}\,dx^m\,dx^n = -e^\xi\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\varphi^2 + e^\eta\,(c\,dt)^2, \tag{50}$$

where ξ and η are functions of r and x^0. The absolute value of the determinant $|g_{mn}|$ will be

$$g = e^{\xi+\eta}\,r^4\sin^2\theta.$$

---

[1] The field equations in five-dimensional theory, when α is allowed to vary, have recently been studied in a thesis by THIRY [13], who has derived the equations (34 a) and (39 a) by a method quite different from ours.

The derivatives with respect to r are denoted by ξ', η', . . ., the derivatives with respect to x^0 by $\mathring{\xi}$ and $\mathring{\eta}$.[1] In the static case there appear no time-derivatives.

We find immediately for the tensor F_{rs},

$$F_{14} = -F_{41} = A', \quad F^{14} = -F^{41} = g^{11}\, g^{44}\, F_{14} = -e^{-\xi-\eta} A',$$

all other components being $= 0$. Further,

$$F_{rs} F^{rs} = F_{14} F^{14} + F_{41} F^{41} = -2\, e^{-\xi-\eta} A'^{\,2}.$$

Introducing the values of g_{rs} from (50) into (34), we get[2]

$$P_{00} = \alpha \cdot e^{-\xi} \left[\frac{1}{2} \alpha \beta^2 \cdot e^{-\eta} A'^{\,2} + \frac{1}{2\alpha} e^{\xi} \left(\overset{\circ\circ}{\xi} + \overset{\circ\circ}{\eta} + \frac{1}{2} \mathring{\xi}^2 + \frac{1}{2} \mathring{\eta}^2 \right) - \right.$$
$$\left. - \frac{1}{\sqrt{\alpha}} (\sqrt{\alpha})'' + \frac{1}{2\sqrt{\alpha}} (\sqrt{\alpha})' \left(\xi' - \eta' - \frac{4}{r} \right) \right] \cdot \qquad (51)$$

By introducing (50) into the expressions for P^{m}_{0} and P^{mn}, we must remember, that, even in the static case, the operators ∂_4 cannot be omitted, since they go over into

$$\partial_4 = \frac{\partial}{\partial x^4} - \beta A_4 \frac{\partial}{\partial x^0} \to - \beta A \frac{\partial}{\partial x^0} \cdot$$

It follows then from (39) through a direct calculation,

$$P^1_0 = e^{-\xi} \left[+ \frac{1}{4} \alpha \beta^2 e^{-\eta} A A' (\mathring{\xi} + \mathring{\eta}) - \right.$$
$$\left. - \frac{1}{2} \left(\mathring{\eta}' + \frac{1}{2} \mathring{\eta} \eta' - \frac{1}{2} \mathring{\xi} \eta' - \frac{2}{r} \cdot \mathring{\xi} \right) + \frac{1}{2\sqrt{\alpha}} (\sqrt{\alpha})' \cdot \mathring{\eta} \right] \qquad (52\,\mathrm{a})$$

$$P^2_0 = P^3_0 \equiv 0,$$

$$P^4_0 = \frac{1}{\sqrt{\alpha}} e^{-\xi-\eta} \left[- \frac{1}{2} \beta \cdot \frac{1}{r^2} \frac{\partial}{\partial r} (\alpha^{3/2} r^2 A') + \frac{1}{4} \alpha^{3/2} \beta A' (\xi' + \eta') - \right.$$
$$\left. - \frac{1}{2} \sqrt{\alpha} \beta A e^{\xi} \left(\overset{\circ\circ}{\xi} - \frac{1}{2} \mathring{\xi} \mathring{\eta} + \frac{1}{2} \mathring{\xi}^2 \right) \right], \qquad (52\,\mathrm{b})$$

and from (45),

---

[1] There seems to be no risk for confusion with the index ° used in other connections.

[2] The values of the four-dimensional three-index symbols in the case of spherical symmetry are given explicitly by EDDINGTON, Mathematical Theory of Relativity (1923), § 38, and by TOLMAN, Relativity, Thermodynamics and Cosmology (1934), § 95.

$$P^{11} = e^{-2\xi}\left[\frac{1}{2}\eta'' - \frac{1}{4}\xi'\eta' + \frac{1}{4}\eta'^2 - \frac{1}{r}\xi' + \right.$$
$$+ \frac{1}{2}\alpha\beta^2 e^{-\eta} A'^2 + \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})'' - \frac{1}{2\sqrt{\alpha}}(\sqrt{\alpha})'\cdot\xi' +$$
$$\left. + \frac{1}{2}\beta^2 A^2 e^{\xi-\eta}\left(\overset{\circ\circ}{\xi} - \frac{1}{2}\mathring{\xi}\mathring{\eta} + \frac{1}{2}\mathring{\xi}^2\right) - \frac{1}{2\alpha}e^{\xi}\left(\overset{\circ\circ}{\xi} + \frac{1}{2}\mathring{\xi}\mathring{\eta} + \frac{1}{2}\mathring{\xi}^2\right)\right], \tag{53 a}$$

$$P^{22} = P^{33}\cdot\sin^2\theta = \frac{e^{-\xi}}{r^3}\left[\frac{1}{r} - \frac{1}{2}(\xi' - \eta') - \frac{e^{\xi}}{r} + \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})'\right], \tag{53 b}$$

$$P^{44} = e^{-\xi-\eta}\left[-\frac{1}{2}\eta'' + \frac{1}{4}\xi'\eta' - \frac{1}{4}\eta'^2 - \frac{1}{r}\cdot\eta' - \right.$$
$$- \frac{1}{2}\alpha\beta^2 e^{-\eta} A'^2 - \frac{1}{2\sqrt{\alpha}}(\sqrt{\alpha})'\cdot\eta' +$$
$$\left. + \frac{1}{2}\beta^2 A^2 e^{\xi-\eta}\left(\overset{\circ\circ}{\xi} - \frac{1}{2}\mathring{\xi}\mathring{\eta} + \frac{1}{2}\mathring{\xi}^2\right) + \frac{1}{2\alpha}e^{\xi}\left(\overset{\circ\circ}{\eta} + \frac{1}{2}\mathring{\xi}\mathring{\eta} + \frac{1}{2}\mathring{\eta}^2\right)\right], \tag{53 c}$$

§ 12. The description of matter

The interaction between the general five-dimensional field and matter, i. e. particles of half-integer spin (spinor particles), is described by the rigorous field equations (6),

$$\Gamma^{\mu\nu} = P^{\mu\nu} - \tfrac{1}{2}\gamma^{\mu\nu}P = -\tfrac{1}{2}\beta^2\Theta^{\mu\nu}.$$

The left members were given in § 8, and in order to get some information as to the character of the tensor $\Theta^{\mu\nu}$, we will recollect some features of the treatment of matter in general four-dimensional theory, as developed by Fock [6], Tetrode [7], Veblen [8], Schrödinger [9], Bargmann [10], and others.[1]

The spinor particles are described by means of the Dirac wave equations

$$g^r\left(\frac{\partial\psi}{\partial x^r} - G_r\psi\right) - \frac{m_0 c}{\hbar}\psi = 0, \quad \left(\frac{\partial\tilde{\psi}}{\partial x^r} + \tilde{\psi}G_r\right)g^r + \tilde{\psi}\frac{m_0 c}{\hbar} = 0, \tag{54}$$

where m_0 is the rest-mass of the particle, while $\tilde{\psi}$ is defined by

$$\tilde{\psi} = i\,\psi^*\boldsymbol{a},$$

where $\boldsymbol{a}$ is a matrix[2], which is introduced to ascertain the right reality properties. The matrices g^r satisfy the relations

$$g^r g^s + g^s g^r = 2\,g^{rs}, \quad g_r g_s + g_s g_r = 2\,g_{rs}, \tag{55}$$

---

[1] We follow the treatment by Schrödinger and Bargmann.

[2] α in the notation of Bargmann.

and may be expressed as linear combinations of the non-varying DIRAC matrices $\bar{g}_r$, satisfying

$$\bar{g}_r \bar{g}_s + \bar{g}_s \bar{g}_r = 2\,\delta_{rs},$$

the δ_{rs} being ordinary KRONECKER symbols. The matrix vector G_r, finally, has the function to make the theory invariant against arbitrary S-transformations. The contravariant components of this matrix vector can be written[1]

$$G^r = \frac{1}{8} g^{rs} \left(g^m \cdot \frac{\partial g^n}{\partial x^s} + \frac{\partial g^n}{\partial x^s} \cdot g^m - 2\, g^{ml} g^{nt} G_{l,ts} \right) g_{(mn)}, \qquad (56)$$

with

$$g_{(mn)} = \tfrac{1}{2} \left(g_m g_n - g_n g_m \right),$$

$G_{l,ts}$, as before, being four-dimensional CHRISTOFFEL symbols.

The symmetric energy-momentum tensor becomes

$$T_{mn} = \frac{i \hbar c}{4} \left[\tilde{\psi}\, g_m \frac{\partial \psi}{\partial x^n} - \frac{\partial \tilde{\psi}}{\partial x^n} g_m \psi + \tilde{\psi}\, g_n \frac{\partial \psi}{\partial x^m} - \frac{\partial \tilde{\psi}}{\partial x^m} g_n \psi - \right.$$

$$\left. - \tilde{\psi} \left(g_m G_n + G_n g_m + g_n G_m + G_m g_n \right) \psi \right].$$

§ 13. The five-dimensional tensor $\Theta^{\mu\nu}$

The formalism outlined here will now be taken over into the five-dimensional theory.[2] We introduce the covariant and contravariant components γ_ϱ and γ^ϱ of a matrix vector, satisfying

$$\gamma_\varrho \gamma_\sigma + \gamma_\sigma \gamma_\varrho = 2\,\gamma_{\varrho\sigma}, \quad \gamma^\varrho \gamma^\sigma + \gamma^\sigma \gamma^\varrho = 2\,\gamma^{\varrho\sigma}, \qquad (57)$$

together with a matrix vector Γ^ϱ with the components

$$\Gamma^\varrho = \frac{1}{8} \gamma^{\varrho\sigma} \left(\gamma^\mu \frac{\partial \gamma^\nu}{\partial x^\sigma} + \frac{\partial \gamma^\nu}{\partial x^\sigma} \gamma^\mu - 2\, \gamma^{\mu\lambda} \gamma^{\nu\tau} \Gamma_{\lambda,\tau\sigma} \right) \gamma_{(\mu\nu)} \qquad (58)$$

with

$$\gamma_{(\mu\nu)} = \tfrac{1}{2} \left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu \right).$$

We have then the wave equations[3],

$$\gamma^\varrho \left(\frac{\partial \psi}{\partial x^\varrho} - \Gamma_\varrho \psi \right) - \frac{m_0 c}{\hbar} \psi = 0, \quad \left(\frac{\partial \tilde{\psi}}{\partial x^\varrho} + \tilde{\psi}\, \Gamma_\varrho \right) \gamma^\varrho + \tilde{\psi} \cdot \frac{m_0 c}{\hbar} = 0, \qquad (59)$$

---

[1] This follows from general arguments by BARGMANN. Alternative forms of this expression have been given by the author [11], and by KLEIN [2].

[2] KLEIN [2].

[3] It is a difficult problem to decide, whether the mass-term $\frac{m_0 c}{\hbar} \psi$ is to be retained or not, when we go over from (54) to (59). See KLEIN [3].

and the energy-momentum becomes

$$\Theta_{\mu\nu} = \frac{i\hbar c}{4}\left[\tilde{\psi}\gamma_\mu \frac{\partial \psi}{\partial x^\nu} - \frac{\partial \tilde{\psi}}{\partial x^\nu}\gamma_\mu \psi + \tilde{\psi}\gamma_\nu \frac{\partial \psi}{\partial x^\mu} - \frac{\partial \tilde{\psi}}{\partial x^\mu}\gamma_\nu \psi - \right.$$

$$\left. - \tilde{\psi}\left(\gamma_\mu \Gamma_\nu + \Gamma_\nu \gamma_\mu + \gamma_\nu \Gamma_\mu + \Gamma_\mu \gamma_\nu\right)\psi\right]. \qquad (60)$$

By comparison of (57) with (55), we see that we may put

$$\gamma^r = g^r, \quad \gamma_0 = g_0, \qquad (61\,\mathrm{a})$$

which gives

$$\gamma_r = g_r + \beta A_r g_0, \quad \gamma^0 = g^0 - \beta A_r g^r, \qquad (61\,\mathrm{b})$$

where g_0 and g^0 are matrices anticommuting with all g_r and g^r. If $\bar{g}_0$ is a matrix satisfying[1]

$$\bar{g}_0 \bar{g}_r + \bar{g}_r \bar{g}_0 = 0, \quad \bar{g}_0^2 = 1,$$

we see from (61) and (57), that we ought to have

$$g_0 = \bar{g}_0 \sqrt{\alpha}, \quad g^0 = \bar{g}_0 \cdot \frac{1}{\sqrt{\alpha}}.$$

Introducing (61) into the wave equations (59), we get for the first of these equations,

$$g^0 \frac{\partial \psi}{\partial x^0} + g^r \cdot \partial_r \psi - (g_r \Gamma^r + g^0 \Gamma_0)\psi - \frac{m_0 c}{\hbar}\psi = 0. \qquad (62)$$

It follows from (61), that

$$\gamma_{(mn)} - \beta A_m \gamma_{(0n)} - \beta A_n \gamma_{(m0)} = g_{(mn)}, \quad \gamma_{(m0)} = g_{(m0)},$$

and, thus, from (58),

$$\Gamma^r = \frac{1}{8} g^{rs}\left[\left(g^m \cdot \partial_s g^n + \partial_s g^n \cdot g^m - 2 g^{ml} g^{nt} \tilde{G}_{l,ts}\right) g_{(mn)} - \right.$$

$$\left. - \frac{2}{\alpha} g^{ml} \tilde{\Gamma}_{l,0s} g_{(m0)} - \frac{2}{\alpha} g^{nt} \tilde{\Gamma}_{0,ts} g_{(0n)} - 2\beta g^{ml} \frac{\partial A_l}{\partial x^s} g_{(m0)}\right],$$

or, using (56) and inserting the values of $\tilde{\Gamma}_{l,0s}$ and $\tilde{\Gamma}_{0,ls}$ from (23) and (24).

$$\Gamma^r = \tilde{G}^r + \frac{1}{4\alpha}\left(\alpha\beta F^{mr} + \frac{\partial g^{mr}}{\partial x^0}\right) g_{(m0)}. \qquad (63\,\mathrm{a})$$

---

[1] KLEIN [2] assumes $\bar{g}_0 = \bar{g}_1 \bar{g}_2 \bar{g}_3 \bar{g}_4$. This leads, however, to difficulties as to the invariance of the wave equations against reflexions in space. The possible ways of escaping this difficulty will not be discussed here (cf. KLEIN [3]).

Further,

$$\Gamma_0 = \frac{1}{8}\left(g^m \cdot \frac{\partial g^n}{\partial x^0} + \frac{\partial g^n}{\partial x^0} \cdot g^m + \alpha \beta F^{m n}\right) g_{(m n)} + \frac{1}{2 \alpha} \frac{\partial \alpha}{\partial x^l} g^{m l} g_{(m 0)} . \qquad (63\,\mathrm{b})$$

From (60) we find

$$\Theta_{00} = \frac{i \hbar c}{2}\left[\tilde{\psi} \gamma_0 \frac{\partial \psi}{\partial x^0} - \frac{\partial \tilde{\psi}}{\partial x^0} \gamma_0 \psi - \tilde{\psi} (\gamma_0 \Gamma_0 + \Gamma_0 \gamma_0) \psi\right].$$

The wave-functions ψ ought to be periodic in x^0 with the same period as the fundamental tensor $\gamma_{\mu\nu}$,

$$\psi = \overset{(0)}{\psi} + \overset{(1)}{\psi} e^{\frac{i}{\hbar} p_0 x^0} + \overset{(-1)}{\psi} e^{-\frac{i}{\hbar} p_0 x^0} + \cdots,$$

the omitted terms corresponding to multiply charged fields. Thus, neglecting terms proportional to $e^{\frac{i}{\hbar} p_0 x^0}$, which disappear by an integration over x^0,

$$\Theta_{00} = -\frac{i \hbar c}{2}\left[\overset{(\tilde{0})}{\psi} \cdot (g_0 \Gamma_0 + \Gamma_0 g_0) \overset{(0)}{\psi} + \overset{(\tilde{1})}{\psi}\left(-\frac{2 i p_0}{\hbar} g_0 + g_0 \Gamma_0 + \Gamma_0 g_0\right) \overset{(1)}{\psi} + \right.$$

$$\left. + \overset{(\widetilde{-1})}{\psi}\left(\frac{2 i p_0}{\hbar} g_0 + g_0 \Gamma_0 + \Gamma_0 g_0\right) \overset{(-1)}{\psi}\right]. \qquad (64\,\mathrm{a})$$

Further,

$$\Theta_0^m = \frac{i \hbar c}{4}\left[\tilde{\psi} g^m \frac{\partial \psi}{\partial x^0} - \frac{\partial \tilde{\psi}}{\partial x^0} g^m \psi + g^{m r} (\tilde{\psi} g_0 \partial_r \psi - \partial_r \tilde{\psi} \cdot g_0 \psi) - \right.$$

$$\left. - \tilde{\psi} (g^m \Gamma_0 + \Gamma_0 g^m + g_0 \Gamma^m + \Gamma^m g_0) \psi\right], \qquad (64\,\mathrm{b})$$

and, if $T_{m n}$ are the covariant components of a tensor, defined by[1]

$$T^{m n} = \Theta^{m n},$$

we have

$$T_{m n} = \frac{i \hbar c}{4}[\tilde{\psi} g_m \partial_n \psi - \partial_n \tilde{\psi} \cdot g_m \psi + \tilde{\psi} g_n \partial_m \psi - \partial_m \tilde{\psi} \cdot g_n \psi -$$

$$- \tilde{\psi} (g_m \tilde{\Gamma}_n + \tilde{\Gamma}_n g_m + g_n \tilde{\Gamma}_m + \tilde{\Gamma}_m g_n) \psi], \qquad (64\,\mathrm{c})$$

where the notations

$$\tilde{\Gamma}_n = \Gamma_n - \beta A_n \Gamma_0$$

are introduced in analogy to the symbols $\tilde{\Gamma}_{r, 0 s}$, $\tilde{\Gamma}_{0, r s}$ and $\tilde{\Gamma}_{m, r s}$ in § 4. The energy function, finally, is defined as

$$H = \int T_4^4 \sqrt{g}\, d\mathrm{v}.$$

---

[1] T_{mn} as defined here differs from the energy-momentum tensor at the end of § 12, since it includes also interaction terms.

PART II.

The linear approximation

Introduction

In the first part of this paper we have dealt with the rigorous equations of the generalized five-dimensional theory, disregarding, however, the problems connected with quantization. Due to their strong non-linearity it is extremely difficult not only to obtain rigorous solutions of these equations but even to develop a rational approximation method for their treatment. In the absence of such a method, KLEIN [2] has proposed a procedure analogous to that given by EINSTEIN [4] for the treatment of gravitational waves. This procedure is based on the assumption that in problems where one is concerned only with long-range forces (ordinary electro-magnetism and gravitation) γ_{00} may be taken equal to 1, while in problems concerned with the "interior" of particles (rest-mass values and nuclear interactions) γ_{00} may be put equal to a very large, constant value a_0.

Using an auxiliary variable

$$x^5 = x^0 \sqrt{\alpha_0},$$

which in the first case $(\alpha \sim 1)$ is identical with x^0, KLEIN puts

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + \sqrt{\alpha_0}\,\beta\,\varphi_{\mu\nu} = \delta_{\mu\nu} + \varepsilon\,\varphi_{\mu\nu}, \tag{65}$$

where $\delta_{\mu\nu}$ are ordinary KRONECKER symbols, while $\varphi_{\mu\nu}$ are of the dimension of an electrical potential, and assumes that $\varepsilon\,\varphi_{\mu\nu}$ may be treated as small quantities.

Even if this treatment lacks a really rational foundation, it is well suited to illustrate the general character of the theory, and it seemed worth while to make a more thorough investigation along these lines. Thus, in the first place, the introduction of the coordinate system mentioned in the first part, where the γ_{m0} are independent of x^0, can be carried through explicitly, whereby only the φ_{mn} depend on x^0.[1] Moreover, the quantization of the field equations is carried through with results which may, perhaps, be helpful for the quantization of the rigorous equations.

Introducing the assumption (65) into the rigorous field equations for the matterfree case and using the periodicity condition

$$\varphi_{\mu\nu} = \sum_{-\infty}^{+\infty} \varphi_{\mu\nu}^{(Z)}\, e^{2\pi i Z \frac{x^0}{l}} = \sum_{-\infty}^{+\infty} \varphi_{\mu\nu}^{(Z)}\, e^{i Z \varkappa x^5}, \tag{66}$$

where

---

[1] In the example, sketched by KLEIN, l. c. p. 13, such a coordinate system was not introduced, and he happened to consider just the x^0-dependent parts of $\gamma_{m\,0}$.

$$\varkappa = \frac{2\pi}{l\sqrt{\alpha_0}} = \frac{e}{\sqrt{\alpha_0}\,\hbar\,\beta\,c}, \tag{67}$$

KLEIN obtained a system of wave equations for each value of Z with a mass-term corresponding to a restmass $Z\mu_0$,

$$\mu_0 = \frac{e}{\sqrt{\alpha_0}\,\beta\,c^2}.$$

Choosing $\sqrt{\alpha_0} \sim 10^{18}$, μ_0 will be of the order of magnitude of meson masses.

§ 14. The elimination of $\overset{Z}{\varphi}_{\mu 5}\,(Z \neq 0)$

We shall now go over to an investigation of the charged fields. We have shown before (§ 2), that it is always possible to find a coordinate transformation, even in the rigorous theory, which carries over to a coordinate system, in which the tensor components γ_{55} and γ_{r5} are independent of x^5, but we will now perform this transformation explicitly in the case of weak fields.

The periodicity condition, inherent in the FOURIER expansion (66), imports, that the five-dimensional coordinate transformations permitted must also be periodic in x^5,

$$\begin{cases} x^5 = n \cdot x'^5 + \xi^5\,(x'^\mu), \\ x^r = \xi^r\,(x'^\mu), \end{cases}$$

where n is an integer, which can generally be put $= 1$, while the functions ξ^ν are of the form

$$\xi^\nu\,(x'^\mu) = \sum_{Z=-\infty}^{+\infty} \overset{(Z)}{\xi^\nu}\,(x^m)\,e^{iZ\varkappa x^5}. \tag{66 a}$$

By an infinitesimal transformation, belonging to this group,

$$x^\nu = x'^\nu + \varepsilon\,\xi^\nu\,(x'^\mu),$$

the $\varphi_{\mu\nu}$, as defined by (65), i. e.

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + \varepsilon\,\varphi_{\mu\nu},$$

are transformed in the following way, as is seen from the ordinary transformation formula for the tensor $\gamma_{\mu\nu}$,

$$\varphi'_{\mu\nu} = \varphi_{\mu\nu} + \frac{\partial\,\xi^\mu}{\partial\,x^\nu} + \frac{\partial\,\xi^\nu}{\partial\,x^\mu} + \cdots,$$

the terms suppressed being proportional to ε.

Inserting the series (66) and (66 a), we get

$$\sum_Z \overset{(Z)\prime}{\varphi}_{55}\; e^{iZ\varkappa x'^5} = \sum_Z \left(\overset{(Z)}{\varphi}_{55} + 2\,i\,Z\,\varkappa\,\overset{(Z)}{\xi}{}^5\right)\cdot e^{iZ\varkappa x^5},$$

$$\sum_Z \overset{(Z)\prime}{\varphi}_{m5}\, e^{iZ\varkappa x'^5} = \sum_Z \left(\overset{(Z)}{\varphi}_{m5} + \frac{\partial \overset{(Z)}{\xi}{}^5}{\partial x^m} + i\,Z\,\varkappa\,\overset{(Z)}{\xi}{}^m\right)\cdot e^{iZ\varkappa x^5},$$

and

$$\sum_Z \overset{(Z)\prime}{\varphi}_{mn}\, e^{iZ\varkappa x'^5} = \sum_Z \left(\overset{(Z)}{\varphi}_{mn} + \frac{\partial \overset{(Z)}{\xi}{}^m}{\partial x^n} + \frac{\partial \overset{(Z)}{\xi}{}^n}{\partial x^m}\right)\cdot e^{iZ\varkappa x^5},$$

where only first order terms are retained. To the actual approximation,

$$e^{iZ\varkappa x'^5} = e^{iZ\varkappa x^5}.$$

Identifying the coefficients in each member of the equations, and choosing, for each value of $Z \neq 0$,

$$\overset{(Z)}{\xi}{}^5 = -\frac{1}{2\,i\,Z\,\varkappa}\,\overset{(Z)}{\varphi}_{55},$$

$$\overset{(Z)}{\xi}{}^m = -\frac{1}{i\,Z\,\varkappa}\left(\overset{(Z)}{\varphi}_{m5} + \frac{\partial \overset{(Z)}{\xi}{}^5}{\partial x^m}\right) = -\frac{1}{i\,Z\,\varkappa}\,\overset{(Z)}{\varphi}_{m5} - \frac{1}{Z^2\varkappa^2}\,\frac{\partial \overset{(Z)}{\varphi}_{55}}{\partial x^m},$$

we see immediately that we bring about that

$$\overset{(Z)\prime}{\varphi}_{55} = \overset{(Z)\prime}{\varphi}_{m5} = 0 \text{ for } Z \neq 0.$$

The coordinate transformations afterwards permitted are, of course, only those, for which all $\overset{(Z)}{\xi}{}^\nu = 0$, if $Z \neq 0$, corresponding to ordinary gauge transformations (11),

$$\begin{cases} x^5 = x'^5 + \overset{(0)}{\xi}{}^5\,(x'^m) \\ x^r = \overset{(0)}{\xi}{}^r\,(x'^m). \end{cases}$$

§ 15. **The independent components**

Introducing (65) into the field equations[1]

$$P_{\mu\nu} = \frac{1}{2}\gamma^{\varrho\sigma}\left(\frac{\partial^2\gamma_{\varrho\sigma}}{\partial x^\mu\,\partial x^\nu} + \frac{\partial^2\gamma_{\mu\nu}}{\partial x^\varrho\,\partial x^\sigma} - \frac{\partial^2\gamma_{\mu\varrho}}{\partial x^\nu\,\partial x^\sigma} - \frac{\partial^2\gamma_{\nu\sigma}}{\partial x^\mu\,\partial x^\varrho}\right) -$$

$$- \gamma^{\varrho\sigma}\,\Gamma^\lambda_{\mu\varrho}\,\Gamma_{\lambda,\nu\sigma} + \gamma^{\varrho\sigma}\,\Gamma^\lambda_{\mu\nu}\,\Gamma_{\lambda,\varrho\sigma} = 0 \qquad (68)$$

we obtain in the first approximation

---

[1] EDDINGTON, Mathematical theory of relativity, § 34.

$$\pentagon\, \varphi_{\mu\nu} + \frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 \varphi_{\mu\varrho}}{\partial x^\nu \partial x^\varrho} - \frac{\partial^2 \varphi_{\nu\varrho}}{\partial x^\mu \partial x^\varrho} = 0 \tag{69}$$

where $\pentagon$ denotes the five-dimensional LAPLACE operator and

$$\varphi = \varphi_{\mu\mu}.$$

Using now the conditions

$$\frac{\partial \varphi_{55}}{\partial x^5} = \frac{\partial \varphi_{m5}}{\partial x^5} = 0 \tag{70}$$

we shall at first put $\mu = \nu = 5$, then $\mu = m$, $\nu = 5$, and, finally, $\mu = m$, $\nu = n$. In the resulting equations, we put $\overset{(0)}{\varphi}_{55} = 0$, which means that we regard α as a constant[1], equal to α_0. Moreover, we neglect gravitational effects, which means that we put all $\overset{(0)}{\varphi}_{mn} = 0$.

We find then, φ denoting now the four-dimensional spur, $\varphi = \varphi_{mm}$,

$$\frac{\partial^2 \varphi}{\partial x^{5^2}} = 0, \tag{71 a}$$

$$\Box\, \varphi_{m5} + \frac{\partial^2 \varphi}{\partial x^m \partial x^5} - \frac{\partial^2 \varphi_{mr}}{\partial x^5 \partial x^r} - \frac{\partial^2 \varphi_{5r}}{\partial x^m \partial x^r} = 0, \tag{71 b}$$

and

$$\Box\, \varphi_{mn} + \frac{\partial^2 \varphi_{mn}}{\partial x^{5^2}} + \frac{\partial^2 \varphi}{\partial x^m \partial x^n} - \frac{\partial^2 \varphi_{mr}}{\partial x^n \partial x^r} - \frac{\partial^2 \varphi_{nr}}{\partial x^m \partial x^r} = 0. \tag{71 c}$$

By introducing now the FOURIER series (66), we write Φ_{mn} and Φ^*_{mn} instead of $\overset{(1)}{\varphi}_{mn}$ and $\overset{(-1)}{\varphi}_{mn}$, the series thus becoming ($\overset{(0)}{\varphi}_{mn} = 0!$),

$$\varphi_{mn} = \frac{1}{\sqrt{2}} \Phi_{mn}\, e^{i\varkappa x^5} + \frac{1}{\sqrt{2}} \Phi^*_{mn}\, e^{-i\varkappa x^5} + \cdots, \tag{72}$$

the omitted terms belonging to multiply charged fields, which will be disregarded here. The factors $\frac{1}{\sqrt{2}}$ are inserted for normalization only, and the connection between Φ_{mn} and Φ^*_{mn} is as follows,

$$\begin{cases} \Phi^*_{mn} = \text{complex conjugate to } \Phi_{mn}, \text{ if } m \neq 4,\ n \neq 4, \text{ or } m = n = 4, \\ \Phi^*_{mn} = \text{complex conjugate to } \Phi_{mn} \text{ with reversed sign, if } m \neq 4,\ n = 4, \end{cases}$$

in order that φ_{mn} in (72), and consequently γ_{mn}, shall have the right reality properties.

---

[1] This is not very satisfying, and makes the physical interpretation of the linear approximation dubious. Our knowledge about the behaviour of α is, however, too scanty for the drawing of any definite conclusions. The equations (69) give $\Box\, \alpha = 0$.

The terms not containing $e^{i\varkappa x^5}$ in (71) give the four ordinary equations for an electro-magnetic field,

$$\Box A_m - \frac{\partial^2 A_r}{\partial x^m \partial x^r} = 0,$$

while the terms proportional to $e^{i\varkappa x^5}$ give rise to ten wave equations,

$$\Box \Phi_{mn} - \varkappa^2 \Phi_{mn} = 0, \tag{73}$$

together with the five subsidiary conditions

$$\Phi = 0, \tag{73 a}$$

$$\frac{\partial \Phi_{mr}}{\partial x^r} = 0. \tag{73 b}$$

This means, that there are only five independent components, corresponding to the five possible spin directions of a particle of spin $2\hbar$. The equations (73), together with (73 a) and (73 b), are, in fact, exactly the equations, given by FIERZ [5] as characteristic of a meson field of spin $2\hbar$.

Introducing the series (66) into the LAGRANGIAN (5), separating the index 5 from the other indices, and using the conditions (70), together with

$$\overset{(0)}{\varphi}_{55} = \overset{(0)}{\varphi}_{mn} = 0,$$

we get

$$\Lambda = \frac{\varepsilon^2}{4}(L_1 + L_2), \tag{74}$$

where

$$L_1 = -\left(\frac{\partial \overset{(0)}{\varphi}_{m5}}{\partial x^n} - \frac{\partial \overset{(0)}{\varphi}_{n5}}{\partial x^m}\right)^2 = -\left(\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m}\right)^2 = -F_{mn}^2 \tag{74 a}$$

is the ordinary LAGRANGIAN for the electro-magnetic field, and

$$L_2 = \frac{\partial \Phi_{mn}^*}{\partial x^r}\left(-\frac{\partial \Phi_{mn}}{\partial x^r} + \frac{\partial \Phi_{mr}}{\partial x^n} + \frac{\partial \Phi_{nr}}{\partial x^m}\right) - \frac{\partial \Phi^*}{\partial x^r}\frac{\partial \Phi_{rm}}{\partial x^m} - \frac{\partial \Phi_{rm}^*}{\partial x^m}\frac{\partial \Phi}{\partial x^r} + \frac{\partial \Phi^*}{\partial x^r}\frac{\partial \Phi}{\partial x^r} - \varkappa^2 \Phi_{mn}^* \Phi_{mn} + \varkappa^2 \Phi^* \Phi \tag{74 b}$$

is the LAGRANGIAN, describing meson fields of spin $2\hbar$, according to the theory of FIERZ. The LAGRANGIAN is given in exactly this form by WENTZEL [12], p. 194, who derives the equations (73), (73 a), and (73 b) by the variation principle.

It may be remarked, that in (74) only those terms are retained, which do not contain any factor $e^{i\varkappa x^5}$, since the other terms disappear by the integration over x^5, which is to be performed by the variation.

§ 16. The second approximation

We shall now examine the next higher approximation.[1] As before, the indices $\mu, \nu, \ldots$ are going from 1 to 5, the coordinate x^5 being defined by $x^5 = x^0 \sqrt{a_0}$. We assume, as before,

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + \varepsilon \varphi_{\mu\nu}, \tag{65}$$

the second term being small as compared to unity. Introducing the FOURIER series (66), and paying regard to the next approximation in performing an infinitesimal coordinate transformation as in § 14, it can be shown without difficulty, that it is possible to bring about that the conditions (70) are true even in this approximation.

Introducing (65), together with the corresponding contravariant expressions, obtained by means of the orthogonality relations for $\gamma_{\mu\nu}$,

$$\gamma^{\mu\nu} = \delta_{\mu\nu} - \varepsilon \varphi_{\mu\nu},$$

into the field equations (68),

$$P_{\mu\nu} = \frac{1}{2} \gamma^{\varrho\sigma} \left(\frac{\partial^2 \gamma_{\varrho\sigma}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 \gamma_{\mu\nu}}{\partial x^\varrho \partial x^\sigma} - \frac{\partial^2 \gamma_{\mu\varrho}}{\partial x^\nu d x^\sigma} - \frac{\partial^2 \gamma_{\nu\sigma}}{\partial x^\mu \partial x^\varrho} \right) -$$
$$- \gamma^{\varrho\sigma} \Gamma^\lambda_{\mu\varrho} \Gamma_{\lambda,\nu\sigma} + \gamma^{\varrho\sigma} \Gamma^\lambda_{\mu\nu} \Gamma_{\lambda,\varrho\sigma} = 0,$$

and suppressing, in the actual approximation, all terms containing higher powers of ε than the second, we can write $P_{\mu\nu}$ in the form

$$P_{\mu\nu} = \overset{(0)}{P}_{\mu\nu} + \overset{(1)}{P}_{\mu\nu} e^{i\varkappa x^5} + \overset{(-1)}{P}_{\mu\nu} e^{-i\varkappa x^5} + \cdots.$$

The field equations are then equivalent to

$$\overset{(0)}{P}_{\mu\nu} = 0, \quad \overset{(1)}{P}_{\mu\nu} = 0, \quad \overset{(-1)}{P}_{\mu\nu} = 0, \ldots,$$

and we will now write down these equations, first for $\mu = \nu = 5$, then for $\mu = m, \nu = 5$, and, finally, for $\mu = m, \nu = n$. The equations $\overset{(-1)}{P}_{\mu\nu} = 0$ are obtained by merely taking the complex conjugate of $\overset{(1)}{P}_{\mu\nu} = 0$, and the equations

$$\overset{(0)}{P}_{55} = 0, \quad \overset{(0)}{P}_{mn} = 0$$

are left out, by the same reasons as before (giving only the equations determining $\overset{(0)}{\varphi}_{55}$ and $\overset{(0)}{\varphi}_{mn}$).

Remembering that $\dfrac{\partial \varphi_{55}}{\partial x^5} = \dfrac{\partial \varphi_{m5}}{\partial x^5} = 0$, we get

$$P_{55} = \frac{\varepsilon}{2} (\delta_{rs} - \varepsilon \varphi_{rs}) \frac{\partial^2 \varphi_{rs}}{\partial x^{5^2}} - \frac{\varepsilon^2}{4} \left(\frac{\partial \varphi_{rs}}{\partial x^5} + \frac{\partial \varphi_{r5}}{\partial x^s} - \frac{\partial \varphi_{s5}}{\partial x^r} \right)^2 = 0,$$

---

[1] This is, of course, still more dubious than the first approximation, It is, however, remarkable, that the second approximation gives a very good agreement with current four-dimensional theory in describing the behaviour of a particle of spin $2\hbar$ in an electromagnetic field.

from which follows

$$\overset{(1)}{P}_{55} = \frac{\varepsilon}{2}\frac{\partial^2 \varphi}{\partial x^{5^2}} - \frac{\varepsilon^2}{2}\frac{\partial \varphi_{rs}}{\partial x^5}\cdot F_{sr} = -\frac{\varepsilon}{2}\varkappa^2 \Phi = 0,$$

the terms in ε^2 disappearing. The condition

$$\Phi = 0 \tag{75}$$

is, therefore, true even in the *second* approximation.

Further,

$$P^m_5 = \frac{\varepsilon}{2}(\delta_{mn} - \varepsilon\varphi_{mn})(\delta_{rs} - \varepsilon\varphi_{rs})\left(\frac{\partial^2\varphi_{rs}}{\partial x^5\partial x^n} + \frac{\partial^2\varphi_{5n}}{\partial x^r\partial x^s} - \frac{\partial^2\varphi_{5r}}{\partial x^n\partial x^s} - \frac{\partial^2\varphi_{ns}}{\partial x^5\partial x^r}\right) -$$

$$-\frac{\varepsilon^2}{2}\varphi_{m5}\frac{\partial^2\varphi}{\partial x^{5^2}} + \frac{\varepsilon^2}{2}\varphi_{5s}\frac{\partial^2\varphi_{ns}}{\partial x^{5^2}} - \frac{\varepsilon^2}{4}\left(\frac{\partial\varphi_{rl}}{\partial x^m} + \frac{\partial\varphi_{ml}}{\partial x^r} - \frac{\partial\varphi_{mr}}{\partial x^l}\right)\cdot$$

$$\cdot\left(\frac{\partial\varphi_{rl}}{\partial x^5} + \frac{\partial\varphi_{5l}}{\partial x^r} - \frac{\partial\varphi_{5r}}{\partial x^l}\right) + \frac{\varepsilon^2}{4}\left(\frac{\partial\varphi_{lm}}{\partial x^5} + \frac{\partial\varphi_{l5}}{\partial x^m} - \frac{\partial\varphi_{m5}}{\partial x^l}\right)\left(2\frac{\partial\varphi_{lr}}{\partial x^r} - \frac{\partial\varphi}{\partial x^l}\right) = 0.$$

Inserting $\overset{(0)}{\varphi}_{m5} = A_m$, and remembering that in the *first* approximation

$$\frac{\partial\Phi_{mr}}{\partial x^r} = 0,$$

we get

$$\overset{(0)}{P}{}^m_5 = \frac{\varepsilon}{2}\left(\Box A_m - \frac{\partial^2 A_r}{\partial x^m\partial x^r}\right) - \frac{\varepsilon^2}{4}\cdot i\varkappa\left[\Phi^*_{rs}\left(\frac{\partial\Phi_{rs}}{\partial x^m} - \frac{\partial\Phi_{ms}}{\partial x^r}\right) -\right.$$

$$\left. -\left(\frac{\partial\Phi^*_{rs}}{\partial x^m} - \frac{\partial\Phi^*_{ms}}{\partial x^r}\right)\Phi_{rs}\right] - \frac{\varepsilon^2}{8}\cdot i\varkappa\left(\frac{\partial\Phi^*_{rs}}{\partial x^m}\Phi_{rs} - \Phi^*_{rs}\frac{\partial\Phi_{rs}}{\partial x^m}\right) = 0,$$

and, suppressing a factor $\frac{1}{\sqrt{2}}$, arising from the normalization in (72),

$$\overset{(1)}{P}{}^m_5 = -\frac{\varepsilon}{2}i\varkappa\cdot\frac{\partial\Phi_{mr}}{\partial x^r} - \frac{\varepsilon^2}{2}\Phi_{rs}\left(\frac{\partial^2 A_m}{\partial x^r\partial x^s} - \frac{\partial^2 A_r}{\partial x^m\partial x^s}\right) - \frac{\varepsilon^2}{2}\Phi_{mn}\left(\Box A_n - \frac{\partial^2 A_r}{\partial x^n\partial x^r}\right) -$$

$$-\frac{\varepsilon^2}{2}A_s\varkappa^2\cdot\Phi_{ms} - \frac{\varepsilon^2}{4}\left(\frac{\partial\Phi_{ms}}{\partial x^r} - \frac{\partial\Phi_{mr}}{\partial x^s}\right)\left(\frac{\partial A_s}{\partial x^r} - \frac{\partial A_r}{\partial x^s}\right) = 0.$$

The last one of these two equations can be written[1]

$$\frac{\partial\Phi_{mr}}{\partial x^r} - \frac{ie}{\hbar c}A_r\Phi_{mr} = \frac{ie}{\hbar c\varkappa^2}\left(\Phi_{rs}\frac{\partial F_{rm}}{\partial x^s} - F_{rs}\frac{\partial\Phi_{rm}}{\partial x^s}\right), \tag{76}$$

---

[1] (76) is the exact counterpart of the corresponding divergence condition in the ordinary theory of *vector* mesons in an electro-magnetic field (see e.g. WENTZEL [12], p. 86).

which replaces the previous divergence condition. Here we have inserted

$$\varepsilon\,\varkappa = \sqrt{a_0}\,\beta \cdot \frac{e}{\sqrt{a_0}\,\hbar\,\beta\,c} = \frac{e}{\hbar\,c}. \tag{67}$$

The equation $\overset{(0)}{P}{}^{m}_{5} = 0$, on the other hand, is of the form

$$\frac{\partial F_{mr}}{\partial x^r} = s_m,$$

where

$$s_m = -\frac{1}{4}\,\frac{i\,e}{\hbar\,c}\left[\left(\frac{\partial \Phi^*_{rs}}{\partial x^m} - 2\,\frac{\partial \Phi^*_{ms}}{\partial x^r}\right)\Phi_{rs} - \Phi^*_{rs}\left(\frac{\partial \Phi_{rs}}{\partial x^m} - 2\,\frac{\partial \Phi_{ms}}{\partial x^r}\right)\right]$$

is the stream vector, formed from the LAGRANGIAN L_2 (74) by the usual rule,

$$s_m = -\frac{i\,e}{\hbar\,c}\left(\frac{\partial L_2}{\partial \frac{\partial \Phi_{rs}}{\partial x^m}}\,\Phi_{rs} - \Phi^*_{rs}\,\frac{\partial L_2}{\partial \frac{\partial \Phi^*_{rs}}{\partial x^m}}\right).$$

Finally, by a simple, but somewhat lengthy, calculation, we get the wave equations

$$\begin{aligned}\overset{(1)}{P}{}^{mn} = \frac{\varepsilon}{2}\left(\Box\,\Phi_{mn} - \varkappa^2\,\Phi_{mn} - \frac{\partial^2 \Phi_{mr}}{\partial x^n\,\partial x^r} - \frac{\partial^2 \Phi_{nr}}{\partial x^m\,\partial x^r}\right) - \\ - \frac{\varepsilon^2}{2}\cdot i\,\varkappa\,A_r\left(2\,\frac{\partial \Phi_{mn}}{\partial x^r} - \frac{\partial \Phi_{mr}}{\partial x^n} - \frac{\partial \Phi_{nr}}{\partial x^m}\right) + \frac{\varepsilon^2}{2}\cdot i\,\varkappa\left(\frac{\partial A_m}{\partial x^r}\,\Phi_{nr} + \frac{\partial A_n}{\partial x^r}\,\Phi_{mr}\right) = 0.\end{aligned} \tag{77}$$

These ten equations together with the five conditions (75) and (76) describe particles of spin $2\,\hbar$ in interaction with an electro-magnetic field, as will be shown in the alternative treatment below.[1]

§ 17. Alternative treatment

In the fore-going paragraph we have started from the five-dimensional LAGRANGIAN Λ, and have got the field equations by varying with respect to the fifteen quantities $\gamma_{\mu\nu}$ (or $\varphi_{\mu\nu}$). The wave equations and the subsidiary conditions have then been obtained as special cases of these general equations by introducing the relations (70),

$$\frac{\partial \varphi_{55}}{\partial x^5} = \frac{\partial \varphi_{m5}}{\partial x^5} = 0.$$

Another possibility is to insert these relations, from the beginning, into the LAGRANGIAN, which is thereby divided into various parts, depending on the four-dimensional quantities A_m, Φ_{mn}, and is then varied with respect to these quantities.

---

[1] We have neglected those terms, which correspond to the interaction with multiply charged fields.

We introduce[1] $\varphi_{m5} = A_m$ and $\varphi_{mn} = \frac{1}{\sqrt{2}} \Phi_{mn} e^{i\varkappa x^5} + \frac{1}{\sqrt{2}} \Phi^*_{mn} e^{-i\varkappa x^5}$, together with the conditions (70), into the LAGRANGIAN density $\Lambda \sqrt{\gamma}$, with Λ given by (5), while the determinant γ has the value

$$\gamma = |\gamma_{\mu\nu}| = 1 + \varepsilon\varphi + \cdots, \text{ or } \sqrt{\gamma} = 1 + \frac{\varepsilon}{2}\varphi + \cdots,$$

which follows immediately from the assumption

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + \varepsilon\varphi_{\mu\nu}.$$

We find then

$$\Lambda\sqrt{\gamma} = \frac{\varepsilon^2}{4}(L_1 + L_2 + L_3) = \frac{\varepsilon^2}{4} L^{\text{total}}, \tag{78}$$

with L_1 and L_2 given by (74), representing the LAGRANGIANS of the electromagnetic field and of the field of spin $2\hbar$, while the interaction term L_3 is

$$L_3 = -\frac{ie}{\hbar c} A_r \left[\Phi^*_{mn} \left(\frac{\partial \Phi_{mn}}{\partial x^r} - 2\frac{\partial \Phi_{mr}}{\partial x^n} \right) - \left(\frac{\partial \Phi^*_{mn}}{\partial x^r} - 2\frac{\partial \Phi^*_{mr}}{\partial x^n} \right) \Phi_{mn} + \right.$$
$$\left. + \Phi^* \left(\frac{\partial \Phi_{mr}}{\partial x^m} - \frac{\partial \Phi}{\partial x^r} \right) - \left(\frac{\partial \Phi^*_{mr}}{\partial x^m} - \frac{\partial \Phi^*}{\partial x^r} \right) \Phi + \Phi^*_{mr} \frac{\partial \Phi}{\partial x^m} - \frac{\partial \Phi^*}{\partial x^m} \Phi_{mr} \right],$$

where we have made use of (67), i.e. $\varepsilon\varkappa = \frac{e}{\hbar c}\cdot$ Other terms in $\Lambda\sqrt{\gamma}$ may be left out, since they disappear by the integration over x^5.

It is not difficult to verify, that $L_2 + L_3$ is obtained from L_2 by changing the operator

$$\frac{\partial}{\partial x^r} \text{ to } \frac{\partial}{\partial x^r} - \frac{ie}{\hbar c} A_r.$$

L^{total} in (78) is, therefore, just the LAGRANGIAN to be used in four-dimensional theory by describing the interaction between a field of spin $2\hbar$ and an electromagnetic field. The field equations

$$\frac{\partial L^{\text{total}}}{\partial \Phi^*_{mn}} - \frac{\partial}{\partial x^r} \frac{\partial L^{\text{total}}}{\partial \frac{\partial \Phi^*_{mn}}{\partial x^r}} = 0$$

become, if we remember, that Φ and $\frac{\partial \Phi_{mr}}{\partial x^r}$ are $= 0$ in the *first* approximation,

$$-\varkappa^2 \Phi_{mn} + \Box \Phi_{mn} + \frac{\partial^2 \Phi}{\partial x^m \partial x^n} - \frac{\partial^2 \Phi_{mr}}{\partial x^n \partial x^r} - \frac{\partial^2 \Phi_{nr}}{\partial x^m \partial x^r} -$$
$$- \delta_{mn} \left(-\varkappa^2 \Phi + \Box \Phi - \frac{\partial^2 \Phi_{rs}}{\partial x^r \partial x^s} \right) -$$

---

[1] As before, we assume $\overset{(0)}{\varphi}_{55} = \overset{(0)}{\varphi}_{mn} = 0$.

$$-\frac{ie}{\hbar c}\Big[A_r\Big(2\frac{\partial \Phi_{mn}}{\partial x^r}-\frac{\partial \Phi_{mr}}{\partial x^n}-\frac{\partial \Phi_{nr}}{\partial x^m}\Big)-$$
$$-\Big(\frac{\partial A_m}{\partial x^r}\Phi_{nr}+\frac{\partial A_n}{\partial x^r}\Phi_{mr}\Big)+\delta_{mn}\frac{\partial A_r}{\partial x_s}\Phi_{rs}\Big]=0. \tag{79}$$

As in the corresponding treatment in the absence of electromagnetic fields (WENTZEL [12], p. 195) we use now, in their proper turns, the operators

$$\sum \delta_{mn},\quad \sum \frac{\partial}{\partial x^n},\quad \sum \frac{\partial^2}{\partial x^m \partial x^n},$$

and employ the fact, following from (73), that

$$\Box\, \Phi_{mn}=\varkappa^2\, \Phi_{mn}$$

to the *first* approximation.

The resulting equations are

$$-2\Box\,\Phi+2\frac{\partial^2 \Phi_{mr}}{\partial x^m \partial x^r}+3\varkappa^2\Phi-2\frac{ie}{\hbar c}\frac{\partial A_r}{\partial x^s}\Phi_{rs}=0, \tag{79 a}$$

$$-\varkappa^2\Big(\frac{\partial \Phi_{mn}}{\partial x^n}-\frac{\partial \Phi}{\partial x^m}\Big)-\frac{ie}{\hbar c}\Big(-\varkappa^2 A_r\Phi_{mr}+\frac{\partial F_{mr}}{\partial x^n}\Phi_{rn}+\frac{\partial \Phi_{mr}}{\partial x^n}F_{rn}\Big)=0, \tag{79 b}$$

and

$$-\varkappa^2\Big(\frac{\partial^2 \Phi_{mn}}{\partial x^m \partial x^n}-\Box\,\Phi\Big)+\frac{ie}{\hbar c}\cdot\varkappa^2\frac{\partial A_r}{\partial x^m}\Phi_{mr}=0. \tag{79 c}$$

From (79 a) and (79 c) it follows immediately, that

$$\Phi=0$$

is true even in the *second* approximation. Inserting this into (79) and (79 b), we get

$$-\varkappa^2\Phi_{mn}+\Box\,\Phi_{mn}-\frac{\partial^2 \Phi_{mr}}{\partial x^n \partial x^r}-\frac{\partial^2 \Phi_{nr}}{\partial x^m \partial x^r}-$$
$$-\frac{ie}{\hbar c}\Big[A_r\Big(2\frac{\partial \Phi_{mn}}{\partial x^r}-\frac{\partial \Phi_{mr}}{\partial x^n}-\frac{\partial \Phi_{nr}}{\partial x^m}\Big)-\Big(\frac{\partial A_m}{\partial x^r}\Phi_{nr}+\frac{\partial A^n}{\partial x^r}\Phi_{mr}\Big)\Big]=0,$$

and

$$\frac{\partial \Phi_{mn}}{\partial x_n}-\frac{ie}{\hbar c}A_n\Phi_{mn}=\frac{ie}{\hbar c\varkappa^2}\Big(\Phi_{rs}\frac{\partial F_{rm}}{\partial x^s}-F_{rs}\frac{\partial \Phi_{rm}}{\partial x^s}\Big),$$

and these equations are identical with (77) and (76). The equivalence between the five-dimensional and the four-dimensional treatment is thereby established.

§ 18. Quantization of the free field of spin $2\hbar$

Before examining, in the linear approximation, the interaction between the general five-dimensional field and the spinor particles, we will examine the properties of the free field of spin $2\hbar$, defined by means of the LAGRANGIAN (74),

$$L_2 = \frac{\partial \Phi^*_{mn}}{\partial x^r}\left(-\frac{\partial \Phi_{mn}}{\partial x^r} + 2\frac{\partial \Phi_{mr}}{\partial x^n}\right) - \frac{\partial \Phi^*}{\partial x^r}\frac{\partial \Phi_{mr}}{\partial x^m} - \frac{\partial \Phi^*_{mr}}{\partial x^m}\frac{\partial \Phi}{\partial x^r} + \frac{\partial \Phi^*}{\partial x^r}\frac{\partial \Phi}{\partial x^r} - \\ - \varkappa^2 \Phi^*_{mn}\Phi_{mn} + \varkappa^2 \Phi^* \Phi,$$

and our nearest task is to perform the quantization of this field. This problem has been solved by FIERZ [5] for arbitrary values of the spin. Here we give a direct explicit calculation in the case of spin $2\hbar$, and we will then show the equivalence with the covariant formulation given by FIERZ.

In deriving the HAMILTONIAN H_2 from (74), we may from the beginning put $\Phi = 0$, for the terms containing Φ in L_2 will make no contribution to the HAMILTONIAN. We separate the index 4 from the other indices, using roman types m, n, r, s, . . ., to denote indices going from 1 to 3, and we get, after rearranging the terms,

$$\begin{aligned} L_2 = &-\frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^4}\frac{\partial \Phi_{\mathrm{mn}}}{\partial x^4} + \frac{\partial \Phi^*_{44}}{\partial x^4}\frac{\partial \Phi_{44}}{\partial x^4} + 2\frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^4}\frac{\partial \Phi_{\mathrm{n}4}}{\partial x^{\mathrm{m}}} + 2\frac{\partial \Phi^*_{\mathrm{n}4}}{\partial x^{\mathrm{m}}}\frac{\partial \Phi_{\mathrm{mn}}}{\partial x^4} + \\ &+ 2\frac{\partial \Phi^*_{\mathrm{m}4}}{\partial x^4}\frac{\partial \Phi_{44}}{\partial x^{\mathrm{m}}} + 2\frac{\partial \Phi^*_{44}}{\partial x^{\mathrm{m}}}\frac{\partial \Phi_{\mathrm{m}4}}{\partial x^4} - \frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{mn}}}{\partial x^{\mathrm{r}}} + 2\frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{mr}}}{\partial x^{\mathrm{n}}} - \\ &- 2\frac{\partial \Phi^*_{\mathrm{m}4}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{m}4}}{\partial x^{\mathrm{r}}} + 2\frac{\partial \Phi^*_{\mathrm{m}4}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{r}4}}{\partial x^{\mathrm{m}}} - \frac{\partial \Phi^*_{44}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{44}}{\partial x^{\mathrm{r}}} - \\ &- \varkappa^2 \Phi^*_{\mathrm{mn}}\Phi_{\mathrm{mn}} - 2\varkappa^2 \Phi^*_{\mathrm{m}4}\Phi_{\mathrm{m}4} - \varkappa^2 \Phi^*_{44}\Phi_{44}. \end{aligned} \tag{80}$$

For H_2, as usual defined as the 44-component of the energy-momentum tensor,

$$H_2 = \frac{\partial L_2}{\partial \dfrac{\partial \Phi_{mn}}{\partial x^4}}\frac{\partial \Phi_{mn}}{\partial x^4} + \frac{\partial \Phi^*_{mn}}{\partial x^4}\frac{\partial L_2}{\partial \dfrac{\partial \Phi^*_{mn}}{\partial x^4}} - L_2,$$

we find then,

$$\begin{aligned} H_2 = &-\frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^4}\frac{\partial \Phi_{\mathrm{mn}}}{\partial x^4} + \frac{\partial \Phi^*_{44}}{\partial x^4}\frac{\partial \Phi_{44}}{\partial x^4} + \frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{mn}}}{\partial x^{\mathrm{r}}} - 2\frac{\partial \Phi^*_{\mathrm{mn}}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{mr}}}{\partial x^{\mathrm{n}}} + \\ &+ 2\frac{\partial \Phi^*_{\mathrm{m}4}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{m}4}}{\partial x^{\mathrm{r}}} - 2\frac{\partial \Phi^*_{\mathrm{m}4}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{\mathrm{r}4}}{\partial x^{\mathrm{m}}} + \frac{\partial \Phi^*_{44}}{\partial x^{\mathrm{r}}}\frac{\partial \Phi_{44}}{\partial x^{\mathrm{r}}} + \\ &+ \varkappa^2 \Phi^*_{\mathrm{mn}}\Phi_{\mathrm{mn}} + 2\varkappa^2 \Phi^*_{\mathrm{m}4}\Phi_{\mathrm{m}4} + \varkappa^2 \Phi^*_{44}\Phi_{44}. \end{aligned}$$

Following the usual method we will now put[1]

---

[1] k is used to denote both the vector (k_1, k_2, k_3) and the absolute value $|k|$. $k \cdot x$ is the scalar product $k_1 \cdot x^1 + k_2 x^2 + k_3 x^3$ (cf. WENTZEL [12].

$$\Phi_{mn} = \frac{1}{\sqrt{V}} \sum_{k=-\infty}^{+\infty} q_{mn}(k) \cdot e^{i k \cdot x} \tag{81 a}$$

and

$$\Phi^*_{mn} = \frac{1}{\sqrt{V}} \sum_{k'=-\infty}^{+\infty} q^*_{mn}(k') \cdot e^{i k' \cdot x}. \tag{81 b}$$

In order to get H_2 expressed only in the five independent components of the tensor Φ_{mn}, we will now eliminate four of the redundant components by introducing the four conditions

$$\frac{\partial \Phi_{mn}}{\partial x^n} = 0, \tag{73 b}$$

and, later on, we will have to get rid of the fifth redundant component by means of the relation $\Phi = 0$.

Introducing the series (81) into (73 b), we find for each value of the momentum vector k,

$$\frac{1}{i c} \dot{q}_{\mathrm{m}4} = - i k_{\mathrm{r}} q_{\mathrm{mr}}, \quad \frac{1}{i c} \dot{q}^*_{\mathrm{m}4} = + i k_{\mathrm{r}} q^*_{\mathrm{mr}}, \tag{82}$$

the argument k being suppressed everywhere (i.e. we write shortly q_{mn} for $q_{mn}(k)$). $\dot{q}_{mn}$ denotes, as usual, the time derivative,

$$\dot{q}_{mn} \equiv \frac{\partial q_{mn}}{\partial t} \equiv i c \frac{\partial q_{mn}}{\partial x^4}.$$

Differentiating (82) with respect to time, and using the wave equation (73), i.e.

$$- \frac{1}{c^2} \ddot{q}_{mn} = k_4^2 q_{mn}, \text{ where } k_4 = + \sqrt{k^2 + \varkappa^2} \quad (k^2 = \sum_{\mathrm{m}=1}^{3} k_{\mathrm{m}}^2).$$

we get

$$q_{\mathrm{m}4} = - \frac{k_{\mathrm{r}}}{c k_4^2} \dot{q}_{\mathrm{mr}}, \qquad q^*_{\mathrm{m}4} = + \frac{k_{\mathrm{r}}}{c k_4^2} \dot{q}^*_{\mathrm{mr}} \tag{83 a}$$

and

$$q_{44} = - \frac{k_{\mathrm{m}} k_{\mathrm{r}}}{k_4^2} q_{\mathrm{mr}}, \quad q^*_{44} = - \frac{k_{\mathrm{m}} k_{\mathrm{r}}}{k_4^2} q^*_{\mathrm{mr}}. \tag{83 b}$$

The series (81) will now be inserted into the expression for H_2. By the integration over the three-dimensional space only terms with $k' = k$ will survive.

$$\int H_2 \, d\mathrm{v} = \sum_k \Big[\frac{1}{c^2} \dot{q}^*_{\mathrm{mn}} \dot{q}_{\mathrm{mn}} - \frac{1}{c^2} \dot{q}^*_{44} \dot{q}_{44} + k_4^2 q^*_{\mathrm{mn}} q_{\mathrm{mn}} - 2 k_{\mathrm{r}} k_{\mathrm{n}} q^*_{\mathrm{mn}} q_{\mathrm{mr}} +$$

$$+ 2 k_4^2 q^*_{\mathrm{m}4} q_{\mathrm{m}4} - 2 k_{\mathrm{r}} k_{\mathrm{m}} q^*_{\mathrm{m}4} q_{\mathrm{r}4} + k_4^2 q^*_{44} q_{44} \Big],$$

the argument k, as before, being suppressed everywhere.

Taking into consideration the expressions (83) for $q_{\mathrm{m}4}$ and q_{44}, we get finally[1],

$$\int H_2\, d\,\mathrm{v} = \sum_k \left(\frac{1}{c^2}\dot{q}^*_{\mathrm{mn}}\,\dot{q}^*_{\mathrm{rs}} + k_4^2\, q^*_{\mathrm{mn}}\, q_{\mathrm{rs}}\right) \text{đ}_{\mathrm{mr}}\, \text{đ}_{\mathrm{ns}}\,, \tag{84}$$

where

$$\text{đ}_{\mathrm{mr}} = \delta_{\mathrm{mr}} - \frac{k_{\mathrm{m}}\, k_{\mathrm{r}}}{k_4^2} \tag{85}$$

is introduced in analogy with the operator

$$d_{\mathrm{mr}} = \delta_{\mathrm{mr}} - \frac{1}{\varkappa^2}\,\frac{\partial}{\partial x^{\mathrm{m}}}\,\frac{\partial}{\partial x^{\mathrm{r}}} = \delta_{\mathrm{mr}} + \frac{k_{\mathrm{m}}\, k_{\mathrm{r}}}{\varkappa^2} \tag{86}$$

used in ordinary quantum field theories. It is easily verified that

$$\text{đ}_{\mathrm{mr}}\, d_{\mathrm{ms}} = \delta_{\mathrm{rs}}\,.$$

In (84) we must bear in mind, that the six quantities q_{mn} are connected by the condition

$$k_4^2\, q_{\mathrm{mm}} = k_{\mathrm{m}}\, k_{\mathrm{r}}\, q_{\mathrm{mr}}\,, \tag{87}$$

which follows from $q_{\mathrm{mm}} = -\,q_{44}$ together with $q_{44} = -\dfrac{k_{\mathrm{m}}\, k_{\mathrm{r}}}{k_4^2}\, q_{\mathrm{mr}}$.

For each term in the series (81), i.e. for each value of k, we choose a system of three mutually perpendicular unit vectors $e_1(k)$, $e_2(k)$, $e_3(k)$, where $e_1(k)$ is parallell to the momentum vector[2],

$$(e_1 \cdot k) = k\,,\ \ (e_2 \cdot k) = (e_3 \cdot k) = 0. \tag{88}$$

For each value of k_1 we have, since (88) implies $k_1 = k$, $k_2 = k_3 = 0$,

$$\begin{aligned} &\text{đ}_{\mathrm{mr}} = 1, \text{ if } \mathrm{m} = \mathrm{r} = 2 \text{ or } \mathrm{m} = \mathrm{r} = 3, \\ &\text{đ}_{\mathrm{mr}} = 1 - \frac{k^2}{k_4^2} = \frac{\varkappa^2}{k_4^2}, \text{ if } \mathrm{m} = \mathrm{r} = 1, \\ &\text{đ}_{\mathrm{mr}} = 0, \text{ if } \mathrm{m} \neq \mathrm{r}. \end{aligned} \tag{89}$$

The HAMILTONIAN goes over into

$$\begin{aligned} \int H_2\, d\,\mathrm{v} = \sum_k \Bigg[&\left(\frac{1}{c^2}\dot{q}^*_{11}\,\dot{q}_{11} + k_4^2\, q^*_{11}\, q_{11}\right) \cdot \frac{\varkappa^4}{k_4^4} + \left(\frac{1}{c^2}\dot{q}^*_{22}\,\dot{q}_{22} + k_4^2\, q^*_{22}\, q_{22}\right) + \\ &+ \left(\frac{1}{c^2}\dot{q}^*_{33}\,\dot{q}_{33} + k_4^2\, q^*_{33}\, q_{33}\right) + 2\left(\frac{1}{c^2}\dot{q}^*_{12}\,\dot{q}_{12} + k_4^2\, q^*_{12}\, q_{12}\right) \cdot \frac{\varkappa^2}{k_4^2} + \\ &+ 2\left(\frac{1}{c^2}\dot{q}^*_{13}\,\dot{q}_{13} + k_4^2\, q^*_{13}\, q_{13}\right) \cdot \frac{\varkappa^2}{k_4^2} + 2\left(\frac{1}{c^2}\dot{q}^*_{23}\,\dot{q}_{23} + k_4^2\, q^*_{23}\, q_{23}\right)\Bigg]. \end{aligned}$$

---

[1] The corresponding expression for *vector* mesons is,

$$\int H_2\, d\,\mathrm{v} = \sum_k \left(\frac{1}{c^2}\dot{q}^*_{\mathrm{m}}\,\dot{q}_{\mathrm{r}} + k_4^2\, q^*_{\mathrm{m}}\, q_{\mathrm{r}}\right) \text{đ}_{\mathrm{mr}}\,.$$

[2] WENTZEL [12], p. 78.

But, from (87), we have

$$k_4^2 (q_{11} + q_{22} + q_{33}) = k^2 q_{11}, \text{ i.e. } q_{11} = -\frac{k_4^2}{\varkappa^2}(q_{22} + q_{33}).$$

This value of q_{11} is inserted into $\int H_2\, d\mathrm{v}$, and the quadratic form is reduced to principal axis by introducing

$$q_{\mathrm{I}} = \sqrt{\frac{3}{2}}(q_{22} + q_{33}), \quad q_{\mathrm{II}} = \sqrt{\frac{1}{2}}(q_{22} - q_{33}). \tag{90 a}$$

In order to facilitate the manner of writing, we put further

$$q_{\mathrm{III}} = \frac{\varkappa\sqrt{2}}{k_4} q_{12}, \quad q_{\mathrm{IV}} = \frac{\varkappa\sqrt{2}}{k_4} q_{13}, \quad q_{\mathrm{V}} = \sqrt{2}\, q_{23}. \tag{90 b}$$

We find then

$$\int H_2\, d\mathrm{v} = \sum_k \sum_{M=I}^{\mathrm{V}} \left(\frac{1}{c^2}\dot{q}_M^* \dot{q}_M + k_4^2 q_M^* q_M\right), \tag{91}$$

i.e. the HAMILTONIAN is written as the sum of five harmonic oscillators.

The quantum conditions are

$$[\dot{q}_M(k), \quad q_{M'}^*(k')] = \frac{\hbar c^2}{i}\delta_{MM'}\delta_{kk'}. \tag{92}$$

The energy is now quantized in the usual way. We put, using the notations of WENTZEL [12],

$$q_M = \sqrt{\frac{\hbar c}{2 k_4}}(a_M + b_M^*), \quad q_M^* = \sqrt{\frac{\hbar c}{2 k_4}}(a_M^* + b_M),$$

$$\dot{q}_M = -i c \sqrt{\frac{\hbar c k_4}{2}}(a_M - b_M^*), \quad \dot{q}_M^* = i c \sqrt{\frac{\hbar c k_4}{2}}(a_M^* - b_M), \tag{93}$$

which means, that the time-dependence is introduced by

$$q_M(k) = \sqrt{\frac{\hbar c}{2 k_4}}(a_M e^{-i c k_4 t} + b_M^* e^{i c k_4 t}),$$

a_M and b_M being matrices, satisfying the commutation relations

$$[a_M(k), \; a_{M'}^*(k')] = [b_M(k), \; b_{M'}^*(k')] = \delta_{MM'}\delta_{kk'},$$

all other brackets being $= 0$. Further,

$$a_M^*(k)\, a_M(k) = N_M^+(k), \quad b_M^*(k)\, b_M(k) = N_M^-(k),$$

and, thus,

$$a_M(k)\, a_M^*(k) = N_M^+(k) + 1, \quad b_M(k)\, b_M^*(k) = N_M^-(k) + 1,$$

where N_M^+ and N_M^- are integers.

The energy becomes,

$$\int H_2 \, d\mathrm{v} = \sum_k \sum_{M=\mathrm{I}}^{\mathrm{V}} \hbar c k_4 (N_M^+ + N_M^-) + H_0 ,$$

where $H_0 = 5 \sum_k \hbar c k_4$ is the vacuum energy.

The electric charge density

$$\varrho = -\frac{i e}{\hbar} \left(\frac{\partial L_2}{\partial \dot{\Phi}_{m n}} \Phi_{m n} - \frac{\partial L_2}{\partial \dot{\Phi}_{m n}^*} \Phi_{m n}^* \right),$$

where e is the elementary charge, becomes from (80), after performing an integration over three-dimensional space and introducing (81) and (83),

$$\int \varrho \, d\mathrm{v} = -\frac{i e}{\hbar c^2} \sum_k (\dot{q}_{\mathrm{mn}}^* q_{\mathrm{rs}} - \dot{q}_{\mathrm{mn}} q_{\mathrm{rs}}^*) \bar{d}_{\mathrm{mr}} \bar{d}_{\mathrm{ns}} .$$

Choosing the coordinate system as before for each value of k, and inserting the values of $\bar{d}_{\mathrm{mr}}$ from (89), we get

$$\int \varrho \, d\mathrm{v} =$$

$$= -\frac{i e}{\hbar c^2} \sum_k \Big[(\dot{q}_{11}^* q_{11} - \dot{q}_{11} q_{11}^*) \cdot \frac{\varkappa_4}{k_4^4} + (\dot{q}_{22}^* q_{22} - \dot{q}_{22} q_{22}^*) + (\dot{q}_{33}^* q_{33} - \dot{q}_{33} q_{33}^*) +$$

$$+ 2 (\dot{q}_{12}^* q_{12} - \dot{q}_{12} q_{12}^*) \frac{\varkappa^2}{k_4^2} + 2 (\dot{q}_{13}^* q_{13} - \dot{q}_{13} q_{13}^*) \cdot \frac{\varkappa^2}{k_4^2} + 2 (\dot{q}_{23}^* q_{23} - \dot{q}_{23} q_{23}^*) \Big],$$

which becomes, by means of (90),

$$\int \varrho \, d\mathrm{v} = -\frac{i e}{\hbar c^2} \sum_k \sum_{M-\mathrm{I}}^{\mathrm{V}} (\dot{q}_M^* q_M - \dot{q}_M q_M^*) = e \sum_k \sum_M (N_M^+ - N_M^-) .$$

Thus, N_M^+ and N_M^- are the number of positively and negatively charged particles respectively, with the spin direction characterized by the index M.

We will now compare the commutation relations (92) with the general covariant expressions, derived by FIERZ,

$$[\Phi_{\mathrm{mn}}(x, t), \Phi_{\mathrm{rs}}^*(x', t')] = \frac{\hbar c^2}{2 i} \left(d_{\mathrm{mr}} d_{\mathrm{ns}} + d_{\mathrm{ms}} d_{\mathrm{nr}} - \frac{2}{3} d_{\mathrm{mn}} d_{\mathrm{rs}} \right) D(x - x', t - t'), \quad (94)$$

where the d_{mr} are given by (86), and the D-function by

$$D(x - x', t - t') = \frac{1}{V} \sum_k e^{i k (x - x')} \cdot \frac{\sin k_4 c (t - t')}{k_4 c} .$$

Differentiating (94) with respect to t and letting $t - t' \to 0$, we get

$$[\dot{\Phi}_{\mathrm{mn}}(x),\ \Phi^*_{\mathrm{rs}}(x')] =$$

$$= \frac{\hbar c^2}{2i}\cdot\frac{1}{V}\sum_k\left((d_{\mathrm{mr}}\,d_{\mathrm{ns}} + d_{\mathrm{ms}}\,d_{\mathrm{n}} - \frac{2}{3}\,d_{\mathrm{mn}}\,d_{\mathrm{rs}}\right)e^{ik(x-x')}. \tag{95}$$

Introducing the series (81) and specializing the coordinate system as in (88), we can set up all the bracket expressions

$$[\dot{q}_{\mathrm{mn}}(k),\ q^*_{\mathrm{rs}}(k')],$$

and, using (90), we find after some elementary calculations the quantum rules (92). Two examples of these calculations will be given here.

From (81) and (94) we find

$$\frac{1}{V}\sum_{kk'}[\dot{q}_{22}(k),\ q^*_{22}(k')]\,e^{i(kx-k'x')} = \frac{\hbar c^2}{2i}\cdot\frac{1}{V}\sum_k\frac{4}{3}\,d_{22}\,d_{22}\,e^{ik(x-x')} = \frac{2}{3}\,\frac{\hbar c^2}{i}\cdot\frac{1}{V}\sum_k e^{ik(x-x')},$$

since $d_{22} = 1$ from (86). Thus,

$$[\dot{q}_{22}(k),\ q^*_{22}(k')] = \frac{2}{3}\,\frac{\hbar c^2}{i}\,\delta_{kk'}.$$

In the same way,

$$[\dot{q}_{33}(k),\ q^*_{33}(k')] = \frac{2}{3}\,\frac{\hbar c^2}{i}\,\delta_{kk'},$$

$$[\dot{q}_{22}(k),\ q^*_{33}(k')] = [\dot{q}_{33}(k),\ q^*_{22}(k')] = -\frac{1}{3}\,\frac{\hbar c^2}{i}\,\delta_{kk'},$$

and, finally,

$$[\dot{q}_{\mathrm{I}}(k),\, q^*_{\mathrm{I}}(k')] = \frac{3}{2}\,[\dot{q}_{22}(k) + \dot{q}_{33}(k),\ q^*_{22}(k') + q^*_{33}(k')] = \frac{\hbar c^2}{i}\,\delta_{kk'}$$

as in (92).

Also,

$$\frac{1}{V}\sum_{kk'}[\dot{q}_{12}(k),\, q^*_{12}(k')]\,e^{i(kx-k'x')} = \frac{\hbar c^2}{2i}\cdot\frac{1}{V}\sum_k\left(d_{11}\,d_{22} + d_{12}\,d_{12} - \frac{2}{3}\,d_{12}\,d_{12}\right)e^{ik(x-x')} =$$

$$= \frac{\hbar c^2}{2i}\cdot\frac{1}{V}\sum_k\frac{k_4^2}{\varkappa^2}\cdot e^{ik(x-x')},$$

since

$$d_{11} = 1 + \frac{k^2}{\varkappa^2} = \frac{k_4^2}{\varkappa^2}, \qquad d_{22} = 1 \text{ and } d_{12} = 0.$$

Thus,

$$[\dot{q}_{\mathrm{II}}(k),\, q^*_{\mathrm{II}}(k')] = \frac{2\varkappa^2}{k_4^2}\,[\dot{q}_{12}(k),\, q^*_{12}(k')] = \frac{2\varkappa^2}{k_4^2}\cdot\frac{\hbar c^2}{2i}\cdot\frac{k_4^2}{\varkappa^2}\,\delta_{kk'} = \frac{\hbar c^2}{i}\,\delta_{kk'}.$$

§ 19. Calculation of the expression $\gamma^\varrho \Gamma_\varrho + \Gamma_\varrho \gamma^\varrho$

The formalism developed in § 13 will now be applied to the linear approximation, and we begin by calculating the expression

$$\gamma^\varrho \Gamma_\varrho + \Gamma_\varrho \gamma^\varrho = g^5 \Gamma_5 + \Gamma_5 g^5 + g_r \Gamma^r + \Gamma^r g_r, \tag{96}$$

which will be used in § 22.

In the formulas (63 a, b) we have then to introduce the assumption

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + \varepsilon \varphi_{\mu\nu},$$

the fifth coordinate x^0 being replaced by $x^5 = x^0 \sqrt{\alpha_0}$. The contravariant components of the fundamental tensor are given by

$$\gamma_{\mu\nu} = \delta_{\mu\nu} - \varepsilon \varphi_{\mu\nu} + \varepsilon^2 \varphi_{\mu\varrho} \varphi_{\nu\varrho},$$

as follows from the rules of orthogonality. This means, that we have, to the actual approximation,

$$g_{mn} = \delta_{mn} + \varepsilon \varphi_{mn} - \varepsilon^2 A_m A_n,$$

$$g^{mn} = \delta_{mn} - \varepsilon \varphi_{mn} + \varepsilon^2 \varphi_{mr} \varphi_{nr} + \varepsilon^2 A_m A_n.$$

The equations

$$g^m g^n + g^n g^m = 2 g^{mn}, \quad g_m g_n + g_n g_m = 2 g_{mn}$$

are then satisfied by

$$g^m = \bar{g}_m - \frac{\varepsilon}{2} \varphi_{mr} \bar{g}_r, \quad g_m = \bar{g}_m + \frac{\varepsilon}{2} \varphi_{mr} \bar{g}_r, \tag{97}$$

$\bar{g}_m$ being ordinary DIRAC matrices[1] (terms in ε^2 may be neglected),

$$\bar{g}_m \bar{g}_n + \bar{g}_n \bar{g}_m = 2 \delta_{mn}.$$

The components of the matrix vector Γ_ϱ are, as follows from (63),

$$\Gamma_5 = \frac{1}{8} \left[\varepsilon F^{mn} + \frac{\varepsilon^2}{4} \left(\varphi_{mr} \frac{\partial \varphi_{nr}}{\partial x^5} - \varphi_{nr} \frac{\partial \varphi_{mr}}{\partial x^5} \right) \right] \cdot g_{(mn)},$$

and

$$\begin{aligned} \Gamma^r = {} & \frac{1}{4} \left[\varepsilon F^{mr} - \varepsilon \frac{\partial \varphi_{mr}}{\partial x^5} + \varepsilon^2 \left(\varphi_{ml} \frac{\partial \varphi_{rl}}{\partial x^5} + \varphi_{rl} \frac{\partial \varphi_{ml}}{\partial x^5} \right) \right] \cdot g_{(m5)} + \\ & + \frac{1}{8} \Big[- \varepsilon (\partial_n \varphi_{mr} - \partial_m \varphi_{nr}) + \frac{\varepsilon^2}{4} (\varphi_{ml} \cdot \partial_r \varphi_{nl} - \varphi_{nl} \cdot \partial_r \varphi_{ml}) + \\ & + \varepsilon^2 \varphi_{ml} (\partial_n \varphi_{rl} - \partial_l \varphi_{nr}) - \varepsilon^2 \varphi_{nl} (\partial_m \varphi_{rl} - \partial_l \varphi_{mr}) + \\ & + \varepsilon^2 \varphi_{rl} (\partial_n \varphi_{ml} - \partial_m \varphi_{nl}) \Big] \cdot g_{(mn)}. \end{aligned}$$

---

[1] $g_5 = \bar{g}_5$ is only an alternative notation for $\bar{g}_0$, since $g_5 = \frac{g_0}{\sqrt{\alpha}}$, and $g_0 = \bar{g}_0 \sqrt{\alpha}$.

We introduce now the relations, following from (97),

$$g^5\, g_{(mn)} + g_{(mn)}\, g^5 = 2\, \bar{g}_5\, \bar{g}_{(mn)} + \varepsilon\, \varphi_{ml}\, \bar{g}_5\, \bar{g}_{(ln)} - \varepsilon\, \varphi_{nl}\, \bar{g}_5\, \bar{g}_{(lm)},$$

$$g_r\, g_{(m5)} + g_{(m5)}\, g_r = 2\, \bar{g}_5\, \bar{g}_{(rm)} + \varepsilon\, \varphi_{rl}\, \bar{g}_5\, \bar{g}_{(lm)} - \varepsilon\, \varphi_{ml}\, \bar{g}_5\, \bar{g}_{(lr)},$$

$$g_r\, g_{(mn)} + g_{(mn)}\, g_r = (\bar{g}_r\, \bar{g}_m\, \bar{g}_n)_{+} + $$
$$+ \frac{\varepsilon}{2}\, \varphi_{rl}\, (\bar{g}_l\, \bar{g}_m\, \bar{g}_n)_{+} + \frac{\varepsilon}{2}\, \varphi_{ml}\, (\bar{g}_r\, \bar{g}_l\, \bar{g}_n)_{+} - \frac{\varepsilon}{2}\, \varphi_{nl}\, (\bar{g}_r\, \bar{g}_l\, \bar{g}_n)_{+},$$

the notation $(\bar{g}_l\, \bar{g}_m\, \bar{g}_n)_{+}$ indicating that the indices l, m, n are all different. Employing the symmetry and antisymmetry properties, we get then

$$g^5\, \Gamma_5 + \Gamma_5\, g^5 = \left[\frac{\varepsilon}{4} F_{mn} - \frac{\varepsilon^2}{4} F_{mr}\, \varphi_{nr} + \frac{\varepsilon^2}{16}\left(\varphi_{mr} \frac{\partial \varphi_{nr}}{\partial x^5} - \varphi_{nr} \frac{\partial \varphi_{mr}}{\partial x^5}\right)\right] \bar{g}_5\, \bar{g}_{(mn)},$$

and

$$g_r\, \Gamma^r + \Gamma^r\, g_r = \left[-\frac{\varepsilon}{2} F_{mn} + \frac{\varepsilon^2}{2} F_{mr}\, \varphi_{nr}\right] \cdot \bar{g}_5\, \bar{g}_{(mn)} + $$
$$+ \frac{\varepsilon^2}{16} (\varphi_{mr} \cdot \partial_l\, \varphi_{nr} - \varphi_{nr} \cdot \partial_l\, \varphi_{mr})\, (\bar{g}_l\, \bar{g}_m\, \bar{g}_n)_{+},$$

where we have used the connection

$$F^{mn} = F_{mn} - \varepsilon\, \varphi_{mr}\, F_{rn} - \varepsilon\, \varphi_{rn}\, F_{mr}.$$

Adding together, inserting the FOURIER series

$$\varphi_{mn} = \frac{1}{\sqrt{2}}\, \Phi_{mn}\, e^{i\varkappa x^5} + \frac{1}{\sqrt{2}}\, \Phi^{*}_{mn}\, e^{-i\varkappa x^5},$$

and neglecting terms proportional to $e^{i\varkappa Z x^5}$ with $|Z| \geqq 2$, we get finally

$$\gamma^{\varrho}\, \Gamma_{\varrho} + \Gamma_{\varrho}\, \gamma^{\varrho} = \left[-\frac{\varepsilon}{4} F_{mn} + \frac{\varepsilon^2}{8} F_{mr}\, (\Phi_{nr}\, e^{i\varkappa x^5} + \Phi^{*}_{nr}\, e^{-i\varkappa x^5}) + \right.$$
$$\left. + \frac{\varepsilon^2 \cdot i\varkappa}{16} (\Phi^{*}_{mr}\, \Phi_{nr} - \Phi_{nr}\, \Phi^{*}_{mr})\right] \bar{g}_5\, \bar{g}_{(mn)} + $$
$$+ \frac{\varepsilon^2}{32} (\Phi^{*}_{mr}\, \partial_l\, \Phi_{nr} + \Phi_{mr}\, \partial_l\, \Phi^{*}_{nr} - \Phi^{*}_{nr} \cdot \partial_l\, \Phi_{mr} - \Phi_{nr}\, \partial_l\, \Phi^{*}_{mr})\, (\bar{g}_l\, \bar{g}_m\, \bar{g}_n)_{+}. \qquad (98)$$

§ 20. The Hamiltonian

Following the method used in the treatment of spinor particles in electromagnetic fields (cf. e.g. WENTZEL [12], § 21), we define the HAMILTONIAN not as the component T_4^4 of the covariant energy-momentum density, but by the expression[1]

$$H = \frac{i\hbar c}{2}\left(\tilde{\psi}\, g^4 \frac{\partial \psi}{\partial x^4} - \frac{\partial \tilde{\psi}}{\partial x^4}\, g^4\, \psi\right). \tag{99}$$

The time-derivatives of ψ and $\tilde{\psi}$ ought to be eliminated by the aid of the wave equations (62), which can be written

$$g^4 \frac{\partial \psi}{\partial x^4} = - g^{\mathrm{r}}\, \partial_{\mathrm{r}} \psi + \varepsilon A_4\, g^4 \frac{\partial \psi}{\partial x^5} - g^5 \frac{\partial \psi}{\partial x^5} + \frac{m_0 c}{\hbar}\psi + (\gamma^{\varrho}\, \Gamma_{\varrho} + \Gamma_{\varrho}\, \gamma^{\varrho})\, \psi,$$

$$\frac{\partial \tilde{\psi}}{\partial x^4}\, g^4 = - \partial_{\mathrm{r}} \tilde{\psi} \cdot g^{\mathrm{r}} + \varepsilon A_4 \frac{\partial \tilde{\psi}}{\partial x^5} \cdot g^4 - \frac{\partial \tilde{\psi}}{\partial x^5}\, g^5 - \tilde{\psi}\, \frac{m_0 c}{\hbar} - \tilde{\psi}\, (\gamma^{\varrho}\, \Gamma_{\varrho} + \Gamma_{\varrho}\, \gamma^{\varrho}).$$

We get then

$$H = \frac{i\hbar c}{2}\Big[\partial_{\mathrm{r}} \tilde{\psi} \cdot g^{\mathrm{r}}\, \psi - \tilde{\psi} \cdot g^{\mathrm{r}}\, \partial_{\mathrm{r}} \psi - \varepsilon A_4 \left(\frac{\partial \tilde{\psi}}{\partial x^5}\, g^4\, \psi - \tilde{\psi}\, g^4 \frac{\partial \psi}{\partial x^5}\right) +$$

$$+ \frac{\partial \tilde{\psi}}{\partial x^5}\, g^5\, \psi - \tilde{\psi}\, g^5 \frac{\partial \psi}{\partial x^5} + 2\, \tilde{\psi}\, \frac{m_0 c}{\hbar}\, \psi + 2\, \tilde{\psi}\, (\gamma^{\varrho}\, \Gamma_{\varrho} + \Gamma_{\varrho}\, \gamma^{\varrho})\, \psi\Big] = H^0 + H', \tag{100}$$

where

$$H^0 = \frac{i\hbar c}{2}\Big[\partial_{\mathrm{r}} \tilde{\psi} \cdot g^{\mathrm{r}}\, \psi - \tilde{\psi}\, g^{\mathrm{r}}\, \partial_{\mathrm{r}} \psi + \frac{2 i e}{\hbar c} A_4 \cdot \sum_Z (Z \cdot \overset{(\tilde{Z})}{\psi} g^4 \overset{(Z)}{\psi}) +$$

$$+ 2 \sum_Z \overset{(\tilde{Z})}{\psi} \left(\frac{m_0 c}{\hbar} - i \varkappa Z g^5\right) \overset{(Z)}{\psi}\Big]$$

corresponds to the ordinary HAMILTONIAN of spinor particles in an electromagnetic field.[2] Here the FOURIER series for ψ and $\tilde{\psi}$ have been inserted together with $\varepsilon \varkappa = \dfrac{e}{\hbar c}$. Those terms have been suppressed, which disappear by the integration over x^5.

The term

$$H' = i\hbar c\, \tilde{\psi}\, (\gamma^{\varrho}\, \Gamma_{\varrho} + \Gamma_{\varrho}\, \gamma^{\varrho})\, \psi,$$

---

[1] H differs from T_4^4 chiefly by terms proportional to A_4 and Γ_4. As well as A_4 may be given an arbitrary value by means of a gauge transformation $A'_m = A_m + \dfrac{\partial \chi}{\partial x^m}$, which does not change the physical quantity $F_{mn} = \dfrac{\partial A_n}{\partial x^m} - \dfrac{\partial A_m}{\partial x^n}$, so may Γ_4 be fixed arbitrarily by an S-transformation $\Gamma'_m = S^{-1}\, \Gamma_m\, S - S^{-1} \dfrac{\partial S}{\partial x^m}$, which does not change the physical quantity $\Phi_{mn} = \left(\dfrac{\partial}{\partial x^m} - \Gamma_m\right) \Gamma_n - \left(\dfrac{\partial}{\partial x^n} - \Gamma_n\right) \Gamma_m = - \dfrac{1}{4} R_{mn,\,rs}\, g^{(rs)} + f_{mn}$ (BARGMANN [10], p. 349).

[2] We must remember, however, that $\tilde{\psi}$ is now defined as $\tilde{\psi} = i\, \psi^*\, a$, where a is a variable matrix, and that g^{r} and g^4 are given by (97). This means that also H^0 contains terms describing interactions with the field of spin $2\hbar$.

where we have to introduce the FOURIER series for ψ and $\tilde{\psi}$ together with the expression for $\gamma^{\varrho} \Gamma_{\varrho} + \Gamma_{\varrho} \gamma^{\varrho}$ from (98), describes interactions of the same type as those characteristic of ordinary meson pair theories.

Stockholms Högskola, Institutet för mekanik och matematisk fysik. February 1951.

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